5.1 Introduction

In previous chapter in depth treatment has been given to simulation and modeling of single two conductors transmission line (one for signal and the other for ground). The model developed will be extended in this chapter to the case of transmission system with more than one signal conductors. Figure 5.1 shows such a multiconductor transmission line system. The layers are assumed to be homogeneous and isotropic.

Figure 5.1 - Cross section of a multiconductor transmission line system

The system consists of n + 1 conductors with conductor 0 taken as ground. For the inhomogeneous case shown electromagnetic fields in the material are not TEM mode. There are bound to have longitudinal components of E and H field to account for the boundary conditions between dielectric and the finite conductivity of the conductors. However for low enough frequencies, the longitudinal EM field is very small in magnitude as compared to the transverse field. Therefore the system is said to approach quasi-TEM modes and the generalized telegraphists’ equation can be used to describe the instantaneous voltage and current relationship in the system. For structures such as microstrip lines and striplines in PCB (printed circuit board), a valid frequency range would be frequency < 10 GHz for quasi-TEM to be valid. This is often beyond the highest harmonics of many digital applications used today.
Discussion on quasi-TEM theory can be found in Frankel 1977, Collins 1992, Lindell and Gu 1987. Assuming quasi-TEM mode of operation, a system of multiconductor transmission line would be described by distributed RLCG elements as depicted in Figure 5.2. The objective of this section is to present a method for modeling a system of coupled multiconductor transmission line system using single uncoupled transmission line models. This method is known as modal analysis and proves to be very effective. It had been investigated by authors such as Tripathi and Rettig 1985, Marx 1973, and Paul 1988 for the lossless case. Subsequently Djordjevic et al 1987 extends modal analysis to lossy multiconductor transmission line system and introduces the impulse response method for analyzing transmission line with non-linear termination. The approach by Djordjevic et al 1987 is used in this chapter. Specifically the writer shows that applicable solutions only exist if the resistance, inductance, capacitance and conductance matrices of the system are realizable (e.g. the matrices are symmetric and positive definite). A general matrix equation similar to equation (4.49) which relates termination voltages and currents of multiconductor transmission line system is also derived. Lossless homogeneous medium transmission line system is treated as a special case and a simulation example will be given at the end of the chapter.

5.2 Wave Equation for Lossy Multiconductor Transmission Line

A distributed schematic of lossy multiconductor transmission lines is depicted in Figure 5.2. The following formulation will be presented in frequency domain. The telegraphists’ equation describing the system in Figure 5.2 is given by (in frequency domain):

\[ \frac{\partial}{\partial z} \left( V(z) \right) = -Z \bar{I}(z) \]  \hspace{1cm} (5.1a)

\[ \frac{\partial}{\partial z} \left( I(z) \right) = -\bar{Y} V(z) \]  \hspace{1cm} (5.1b)
where:
\[
\begin{bmatrix}
V_1(z) \\
V_2(z) \\
\vdots \\
V_n(z)
\end{bmatrix} = \begin{bmatrix}
\bar{V}(z)
\end{bmatrix}, \quad
\begin{bmatrix}
I_1(z) \\
I_2(z) \\
\vdots \\
I_n(z)
\end{bmatrix} = \begin{bmatrix}
\bar{I}(z)
\end{bmatrix}, \quad \text{and}^1
\]

\[
\begin{bmatrix}
12 \\
11 \\
\vdots \\
nn
\end{bmatrix}
\]

\[
\begin{bmatrix}
\omega \\
\omega \\
\vdots \\
\omega
\end{bmatrix}
\]

**Figure 5.2** - Schematics of multiconductor transmission lines

\[
\bar{Z} = \begin{bmatrix}
(R_{11} + j\omega L_{11}) & (R_{12} + j\omega L_{12}) & \cdots & (R_{1n} + j\omega L_{1n}) \\
(R_{21} + j\omega L_{21}) & (R_{22} + j\omega L_{22}) & \cdots & (R_{2n} + j\omega L_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
(R_{n1} + j\omega L_{n1}) & (R_{n2} + j\omega L_{n2}) & \cdots & (R_{nn} + j\omega L_{nn})
\end{bmatrix}
\]

\[
\bar{Y} = \begin{bmatrix}
(G_{11} + j\omega B_{11}) & (G_{12} + j\omega B_{12}) & \cdots & (G_{1n} + j\omega B_{1n}) \\
(G_{21} + j\omega B_{21}) & (G_{22} + j\omega B_{22}) & \cdots & (G_{2n} + j\omega B_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
(G_{n1} + j\omega B_{n1}) & (G_{n2} + j\omega B_{n2}) & \cdots & (G_{nn} + j\omega B_{nn})
\end{bmatrix}
\]

with \(B_{ij} = C_{ij}, \quad i = 1, 2, 3 \ldots n, \quad j = 1, 2, 3 \ldots \). 

B is the symbol used for susceptance. The mutual resistance in equation (5.2a) is due to proximity effect, where distribution of current density is disturbed by magnetic field from adjacent conductors’ current. Under the condition where

---

1 Here the symbol \(\bar{V}\) describes a square matrix while \(\bar{V}\) describes a vector.
cross section of each transmission line is small compared to the wavelength, proximity effect can often be neglected, thus $R_{ij} = 0$ for $i \neq j$. The $n \times n$ matrices $\overline{R}$, $\overline{L}$, $\overline{C}$, $\overline{G}$ are real and symmetric, resulting in symmetrical complex matrices $\overline{Z}$ and $\overline{Y}$. It can be shown in Section 5.6 that the elements in $\overline{G}$ is related to elements in $\overline{C}$ by:

$$
\overline{G}_{ij} = \omega \varepsilon \left( \tan \delta \right) \overline{C}_{ij}
$$

(5.3)

Equation (5.1a) can be written in the familiar wave equation form by differentiating (5.1a) with respect to $z$ and substituting equation (5.1b):

$$
\frac{\partial^2}{\partial z^2} V(z) = \overline{Z} \overline{V}(z)
$$

(5.4a)

Similar operation on equation (5.2b) yields:

$$
\frac{\partial^2}{\partial z^2} I(z) = \overline{Y} \overline{I}(z)
$$

(5.4b)

Let the matrices solution for equation (5.4a) and (5.4b) be of the form:

$$
\overline{V}(z) = \overline{V}_o e^{-\gamma z} \text{ for positive direction traveling wave and }
\overline{V}(z) = \overline{V}_o e^{+\gamma z} \text{ for negative direction traveling wave.}
$$

(5.5a)

Similarly for current:

$$
\overline{I}(z) = \overline{I}_o e^{\pm j \gamma z}
$$

(5.5b)

where: $\gamma = \alpha + j \beta$

(5.72c)

Focusing on positive traveling wave, one would generally write $\overline{V}_o = \overline{V}_o$ and $\overline{I}_o = \overline{I}_o$. Substituting equations (5.5a) and (5.5b) into equations (5.4a) and (5.4b) respectively:

$$
[\overline{Z} - \overline{U} \gamma^2] \overline{V}_o = 0
$$

(5.6a)

$$
[\overline{Y} - \overline{U} \gamma^2] \overline{I}_o = 0
$$

(5.6b)

with
\[
\bar{U} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix} = n \times n \text{ identity matrix} \quad (5.6c)
\]

### 5.3 Modal Solution of Multiconductor Transmission Line

A closer inspection would reveal that the results are a system of eigenvalue problems given by equations (5.6a) and (5.6b), in the form of \([\bar{A} - \bar{U}v]\bar{E} = 0\), where \(\bar{A}\) is the \(n \times n\) matrix, \(v\) = eigenvalue and \(\bar{E}\) the \(n \times 1\) eigenvector. This problem will only have non-trivial solutions if \(|\bar{A} - \bar{U}v| = 0\).² It is interesting to note that equations (5.6a) and (5.6b) have similar eigenvalues because \(\bar{ZY}\) and \(\bar{YZ}\) are the transpose of each other. The solutions to equations (5.6a) and (5.6b) are called modes while the set of corresponding eigenvalues is called the spectrum. In order that the modes represent traveling waves, the eigenvalue \(\gamma^2\) must be complex. A requirement for traveling waves would be both \(\alpha\) and \(\beta\) of \(\gamma\) must be real and positive (\(\gamma = \alpha + j\beta\)).

Let \(\gamma^2 = \lambda_R + j\lambda_I = (\alpha + j\beta)^2\) \quad (5.7a)

with \(\lambda_R = \alpha^2 - \beta^2\) and \(\lambda_I = 2\alpha\beta\) \quad (5.7b)

Solving for \(\alpha\) and \(\beta\) we would arrive at the following relationship since equation (5.7b) will results in quadratic equations:

\[
\alpha^2 = \frac{-\lambda_R \pm \sqrt{\lambda_R^2 + \lambda_I^2}}{2} \quad (5.8a)
\]

\[
\beta^2 = \frac{\lambda_R \pm \sqrt{\lambda_R^2 + \lambda_I^2}}{2} \quad (5.8b)
\]

² The symbol \(| |\) denotes the determinant of matrix
In order to determine the expression that would give the correct $\alpha$ and $\beta$, we would consider the limiting condition when the system is lossless, e.g. $\overline{R} = \overline{G} = \mathbf{0}$. Equation (5.6a) then becomes:

$$\left| -\omega^2 LC - U\gamma^2 \right| = 0$$

(5.8a)

which can be written as:

$$(-1)^n \left| \omega^2 LC - U(-\gamma^2) \right| = 0$$

(5.8b)

The matrix product of realizable $\overline{LC}$ is positive definite since from Marx 1973 it can be written in the form:

$$\overline{LC} = \overline{\lambda} \left( \frac{1}{C^2} \right) \left( \frac{1}{L^2} \right)^T \left( \frac{1}{C^2} \right) \overline{\lambda}^{-1} \left( \frac{1}{L^2} \right)$$

(5.9)

The matrix $\overline{B} = \left( \frac{1}{C^2} \right)^T \left( \frac{1}{L^2} \right)$ is positive definite, meaning it has real and positive eigenvalues (it has the form $\overline{AA}^T$). $\overline{LC}$ is just a similarity transformation of $\overline{B}$. Thus according to matrix theory it must have the same eigenvalues. Under lossless condition:

$$\gamma^2 = -\beta^2 \text{ or } -\gamma^2 = \lambda_R \geq 0 \text{ and } \gamma = j\beta,$$

(5.10a)

$$\beta \geq 0, \alpha = 0$$

(5.10b)

Upon investigating equations (5.8a) and (5.8b), the only expressions that will fulfill condition of equations (5.10a) and (5.10b) are:

$$\alpha = -\frac{1}{2} \left( -\lambda_R + \sqrt{\lambda_R^2 + \lambda_i^2} \right)$$

(5.11a)

and

$$\beta = \frac{1}{2} \left( \lambda_R + \sqrt{\lambda_R^2 + \lambda_i^2} \right)$$

(5.11b)

Thus from equations (5.11a) and (5.11b), it is seen that $\alpha$ and $\beta$ will always be positive or greater than zero regardless of the eigenvalue. Therefore it is proven that a solution will always exist for equations (5.6a) and (5.6b) as long as the equivalent inductance and capacitance matrices of the multiconductor system are
realizable. In this instance realizable requirement will dictate the matrices to become positive definite.

5.4 Network Analysis and Simulation for Lossy Multiconductor Transmission Line

Let \( \overrightarrow{M_V} \) be a voltage modal matrix whose columns consists of \( n \overrightarrow{V}_i \) (\( i = 0,1,2...n \)) eigenvectors for equation (5.6a) and \( \overrightarrow{M_I} \) a current modal matrix whose columns consist of \( n \overrightarrow{I}_j \) vectors for equation (5.6b). For now consider only the positive traveling waves. A complete solution of equations (5.1a) and (5.1b) including the phase terms will be given by :

\[
\begin{bmatrix}
V_{o1} & V_{o2} & \cdots & V_{on}
\end{bmatrix}
\begin{bmatrix}
e^{-\gamma z_1} & 0 & \cdots & 0 \\
0 & e^{-\gamma z_2} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{-\gamma z_n}
\end{bmatrix}
= \overrightarrow{M_V} \text{Diag}[e^{-\gamma z_1}, e^{-\gamma z_2}, \cdots, e^{-\gamma z_n}]
\]

(5.12a)

\[
\begin{bmatrix}
I_{o1} & I_{o2} & \cdots & I_{on}
\end{bmatrix}
\begin{bmatrix}
e^{-\gamma z_1} & 0 & \cdots & 0 \\
0 & e^{-\gamma z_2} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{-\gamma z_n}
\end{bmatrix}
= \overrightarrow{M_I} \text{Diag}[e^{-\gamma z_1}, e^{-\gamma z_2}, \cdots, e^{-\gamma z_n}]
\]

(5.12b)

The voltage and current modal matrices are dependent on each other since they must satisfy equations (5.1a) and (5.1b). Substituting equations (5.12a) and (5.12b) into equations (5.1a) and (5.1b) respectively will result in the following relations :

\[
\overrightarrow{M_V} = \overrightarrow{ZM_I} \overrightarrow{S}
\]

(5.13a)
Therefore to determine the spectrum of solutions for the multiconductor transmission line system, we could embark by solving equation (5.6a) for the voltage modal matrix and proceed to determine the current modal matrix using equation (5.13b). Conversely we might wish to solve for current matrix first and determine voltage matrix subsequently using equation (5.13a). It has been shown by Marx 1973 that if voltage modal matrix and current modal matrix are solved independently using equations (5.6a) and (5.6b), then the matrices are orthogonal with respect to each other. $\overrightarrow{M}_v$ and $\overrightarrow{M}_i$ are non-singular from matrix theory, the $n$ column vectors in these modal matrices span an $n$ dimensional space. Therefore any arbitrary sets of voltage and current wave amplitudes propagating along the multiconductor transmission line can be represented as a linear combination of the column vectors from $\overrightarrow{M}_v$ and $\overrightarrow{M}_i$.

Transmission line voltages and currents at any position along the axis can be written as sums of the incident and reflected waves (Djordjevic 1987). Proceeding along the approach of Section 4.5 for single transmission line, network equation similar to equation (4.49) in frequency domain for termination voltages and currents can be obtained for multiconductor transmission line. Adopting the direction convention of Figure 4.4, incident and reflected voltage waves can be thought of as a superposition of voltage modal waves.

$$\overrightarrow{V}(z) = \overrightarrow{V}_i(z) + \overrightarrow{V}_r(z) = \overrightarrow{M}_v \left[ \overrightarrow{D}(z)\overrightarrow{V}_o + \overrightarrow{D}^{-1}(z)\overrightarrow{V}_o \right]$$

(5.14a)
\( \overline{D}(z) = \begin{bmatrix} e^{-\gamma z} & 0 & \cdots & 0 \\ 0 & e^{-\gamma z} & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\gamma z} \end{bmatrix} \) \hspace{1cm} (5.14b)

\( \overline{V}_o^i = \text{Complex intensities of incident modes} = \begin{bmatrix} V_{o1}^+ \\ V_{o2}^+ \\ \vdots \\ V_{on}^+ \end{bmatrix} \) \hspace{1cm} (5.14c)

\( \overline{V}_o^- = \text{Complex intensities of reflected modes} = \begin{bmatrix} V_{o1}^- \\ V_{o2}^- \\ \vdots \\ V_{on}^- \end{bmatrix} \) \hspace{1cm} (5.14d)

Similar expression can be written for current waves:

\[ I(z) = I_i(z) + I_r(z) = \overline{M}_I \left[ \overline{D}(z) \overline{I}_o^+ - \overline{D}^{-1}(z) \overline{I}_o^- \right] \] \hspace{1cm} (5.15)

The characteristic impedance of the multiconductor transmission line system is defined by:

\[ \overline{Z}_c = \overline{M}_V \overline{M}_I^{-1} \] \hspace{1cm} (5.16a)

Consequently equation (5.15) can be written as:

\[ I(z) = \overline{Z}_c^{-1} \overline{M}_V \left[ \overline{D}(z) \overline{I}_o^+ - \overline{D}^{-1}(z) \overline{I}_o^- \right] \] \hspace{1cm} (5.17)

Applying equations (5.14a) and (5.16b) at the terminals \( z = 0 \) and \( z = L \) of the multiconductor transmission line system. At \( z = 0 \):
\[ V(0) = M_v \left( V_o^+ + V_o^- \right) \]  
\[ Z_c \bar{I}(0) = M_v \left( V_o^+ - V_o^- \right) \]  
At \( z = L \):

\[ V(L) = M_v \left( \bar{D}(L)V_o^+ + \bar{D}^{-1}(L)V_o^- \right) \]  
\[ Z_c \bar{I}(0) = M_v \left( \bar{D}(L)V_o^- - \bar{D}^{-1}(L)V_o^+ \right) \]

Eliminating \( V_o^+ \) and \( V_o^- \) from equations (5.18a) to (5.19b) and after some algebraic manipulation, the following expression will be obtained:

\[ \begin{bmatrix} \bar{U} & -M_v \bar{D}(L)M_v^{-1} \bar{V}(0) \\ -M_v \bar{D}(L)M_v^{-1} \bar{V}(L) \end{bmatrix} = \begin{bmatrix} \bar{Z}_c & -M_v \bar{D}(L)M_v^{-1} \bar{I}(0) \\ -M_v \bar{D}(L)M_v^{-1} \bar{I}(L) \end{bmatrix} \]

(5.20)

Note that in equation (5.20) the elements within the matrices are themselves matrices! When the termination for the multiconductor transmission line are linear, equation (5.20) can be combined with the expressions for termination conditions and the voltages or currents at \( z = 0 \) or \( z = L \) can be solved in the frequency domain. For linear termination equivalent Thevenin or Norton networks can be assigned to the termination as shown in Figure 5.3. The matrix equation for terminating condition can be written as:

\[ \begin{bmatrix} \bar{V}(0) \\ \bar{V}(L) \end{bmatrix} = \begin{bmatrix} \bar{V}_g & 0 \\ 0 & \bar{Z}_g \end{bmatrix} \begin{bmatrix} \bar{I}(0) \\ \bar{I}(L) \end{bmatrix} \]

(5.21)
Equations (5.21) and (5.20) can be solved for either the termination voltage matrices or the current matrices in terms of source voltages at \( z = 0 \) and \( z = L \). For instance solving for the termination voltage matrices, one would obtain after some matrix manipulation:

\[
\begin{bmatrix}
\left( \bar{U} + \bar{Z} \bar{Z}^{-1} \right) & -\bar{M} \cdot D(L) \left( \bar{M} \bar{Z}^{-1} - \bar{M} \bar{Z}^{-1} \right) \\
-\bar{M} \cdot D(L) \left( \bar{M} \bar{Z}^{-1} - \bar{M} \bar{Z}^{-1} \right) & \left( \bar{U} + \bar{Z} \bar{Z}^{-1} \right)
\end{bmatrix}
\begin{bmatrix}
\bar{V}(0) \\
\bar{V}(L)
\end{bmatrix}
= \begin{bmatrix}
\bar{Z} \bar{Z}^{-1} & -\bar{M} \cdot D(L) \bar{M} \bar{Z}^{-1} \\
\bar{M} \cdot D(L) \bar{M} \bar{Z}^{-1} & \bar{Z} \bar{Z}^{-1}
\end{bmatrix}
\begin{bmatrix}
\bar{V}_c \\
\bar{V}
\end{bmatrix}
\tag{5.22}
\]

Equation (5.22) can be used for frequency domain analysis of multiconductor transmission line response with linear termination. Conversion to time domain can be performed using inverse Fourier analysis or inverse Fourier Transform. If periodic stimulation is used, care must be taken to allow sufficient time for the response of the system to die off before subsequent stimulation arrives in time domain. This is to prevent overlapping of responses and will manifest as frequency aliasing in the frequency domain. Under the circumstances when non-linear termination is connected to either ends of the multiconductor transmission line, impulse responses method similar to Section 4.4 has to be used. Time domain response for the termination currents must be determined for each individual case when a unit impulse voltage is applied at one of the terminals while all other terminals are short-circuited. In order to aid the convergence of the impulse response, artificial resistance with value \( R_i \), \( i=1,2,3 \ldots n \) correspond to the diagonal elements of the characteristic impedance matrix \( Z_{cii} \) are inserted between
the transmission line and source/load termination (Djordjevic 1987). The time domain response of current for arbitrary voltage source is then computed using numerical convolution as in equation (4.58). To obtain the actual voltage and current, negative resistance has to be connected in series to the multiconductor transmission line. A model relating the voltage-current characteristics of the multiconductor transmission line can then be created. This voltage-current characteristics of the model would depend on instantaneous voltage/current and past voltage/current values at the terminals.

A summary of the procedures for impulse response method as applied to multiconductor transmission line based on Djordjevic 1987 is given below:

1. Form an augmented system as in Figure 5.4. The nodes between the positive and negative resistance are labeled as the virtual nodes, with voltages $v_{vk}$.

   Equivalent representation of a lossy transmission line with non-linear terminal network.

Using equations (5.20) and (5.21) to solve for the current responses when one voltage source is a unit impulse while all other voltage sources are set to zero. Label the current responses as $i_{gkj}^s$ and $i_{gkj}^0$. The current Green’s function $i_{gkj}^s$ is the current in conductor $k$ due to unit impulse source in conductor $j$, where superscript ‘s’ denotes both the source and response are at the same termination...
end. The current Green’s function \( i_{gkj}^0 \) is the current in conductor \( k \) due to unit impulse source in conductor \( j \), where superscript ‘o’ denotes source and response are at opposite termination ends. Total response at the generator and load ends of the multiconductor transmission line would be given by convolution of the impulse current response and the respective virtual voltage functions.

\[
i_{kG} = \sum_{j=1}^{n} \int_{0}^{t} i_{gkj}^0 (t-\alpha)v_{vjG}(\alpha)d\alpha + \sum_{j=1}^{n} \int_{0}^{t} i_{gkj}^0 (t-\alpha)v_{vjk}(\alpha)d\alpha \tag{5.23a}
\]

\[
i_{kL} = \sum_{j=1}^{n} \int_{0}^{t} i_{gkj}^0 (t-\alpha)v_{vjG}(\alpha)d\alpha + \sum_{j=1}^{n} \int_{0}^{t} i_{gkj}^0 (t-\alpha)v_{vjk}(\alpha)d\alpha \tag{5.23b}
\]

\( k = 1,2,\cdots,n \)

2. Replacing the integration by summation using rectangular rule and noting that \( i_{gkj}^0(0) \neq 0 \) and \( i_{gkj}^0(0) = 0 \) due to the line delay:

\[
i_{kG}(q) = \sum_{j=1}^{n} i_{gkj}^0(0)v_{vjG}(q)\Delta t + \sum_{j=1}^{n} \sum_{p=0}^{q-1} i_{gkj}^0(q-p)v_{vjG}(p)\Delta t + \sum_{j=1}^{n} \sum_{p=0}^{q-1} i_{gkj}^0(q-p)v_{vjk}(p)\Delta t
\]

\[\Rightarrow \tilde{i}_{kG}(q) = \tilde{G}_{vd} v_{vG}(q) + \tilde{i}_{cG}(q-1) \tag{5.24a}\]

\[
i_{kL}(q) = \sum_{j=1}^{n} i_{gkj}^0(0)v_{vjk}(q)\Delta t + \sum_{j=1}^{n} \sum_{p=0}^{q-1} i_{gkj}^0(q-p)v_{vjG}(p)\Delta t + \sum_{j=1}^{n} \sum_{p=0}^{q-1} i_{gkj}^0(q-p)v_{vjk}(p)\Delta t
\]

\[\Rightarrow \tilde{i}_{kL}(q) = \tilde{G}_{vd} v_{vL}(q) + \tilde{i}_{cL}(q-1) \tag{5.24b}\]

\( k = 1,2,\cdots,n \quad q = 1,2,3\cdots \) (discrete time step)

3. Similar to single transmission line case, the double sums in equations (5.24a) and (5.24b) correspond to \( \tilde{i}_{cG}(q-1) \) and \( \tilde{i}_{cL}(q-1) \) while the single sums correspond to \( \tilde{G}_{vd} (\tilde{G}_{vd} \text{ is symmetrical with respect to generator and load ends}) \). Solving for the virtual voltages at \( t = q\Delta t \):
\[
\tilde{v}_G(q) = \tilde{v}_{iG}(q) + \text{diag}[-Z_{c11}, -Z_{c22}, \ldots, -Z_{cmn}] \tilde{i}_G(q)
\]

\[
= \left[ \tilde{G}_{vd}^{-1} + \text{diag}[-Z_{c11}, -Z_{c22}, \ldots, -Z_{cmn}] \right] \tilde{i}_G(q) - \tilde{G}_{vd}^{-1} \tilde{i}_{iG}(q-1)
\]

\[
\tilde{v}_L(q) = \tilde{v}_{iL}(q) + \text{diag}[-Z_{c11}, -Z_{c22}, \ldots, -Z_{cmn}] \tilde{i}_L(q)
\]

\[
= \left[ \tilde{G}_{vd}^{-1} + \text{diag}[-Z_{c11}, -Z_{c22}, \ldots, -Z_{cmn}] \right] \tilde{i}_L(q) - \tilde{G}_{vd}^{-1} \tilde{i}_{iL}(q-1)
\]

Equations (5.25a) and (5.25b) represents the voltage-current relations for the multiconductor transmission line network at discrete time of \(q\Delta t\). The relation can be interpreted as an impedance term relating the instantaneous current to voltage and a voltage term due to past responses of all the terminals. The equations can be viewed graphically as in Figure 5.5.

![Graphical representation of V-I relations lossy multiconductor transmission line](image)

**Figure 5.5** - Graphical representation of V-I relations lossy multiconductor transmission line.

This form is suitable for implementation in SPICE circuit simulator although not exactly convenient. Under the condition when the multiconductor transmission line system with \(n+1\) conductors is lossless, an equivalent circuit consisting of \(n\) non-coupled transmission and two transformation networks at the termination can be constructed. The equivalent circuit is valid for both linear and non-linear termination.
5.5 Lossless Multiconductor Transmission Line as a Special Case

When the multiconductor transmission line system is lossless, the impedance matrix \( \overline{Z} \) becomes \( j\omega \overline{L} \) while the admittance matrix \( \overline{Y} \) becomes \( j\omega \overline{C} \). Unlike the lossy case, both \( \overline{L} \) and \( \overline{C} \) are independent of frequency. The wave equations for multiconductor transmission line can be conveniently written in time domain and an equivalent circuit using linear circuit elements is derived by using suitable linear transformation (Tripathi et al. 1985, Paul 1988). Applying inverse Fourier transform to equations (5.1a) and (5.1b) will results in time domain equations of the form:

\[
\frac{d}{dz} \overline{V}(z,t) = -\overline{L} \frac{d}{dt} \overline{I}(z,t) \tag{5.26a}
\]

\[
\frac{d}{dt} \overline{I}(z,t) = -\overline{C} \frac{d}{dz} \overline{V}(z,t) \tag{5.26b}
\]

5.5.1 Equivalent Circuit of Lossless Multiconductor Transmission Line

Through suitable linear transformation to the voltage and current vectors, the inductance and capacitance matrices can be diagonalized in order to derive an uncoupled wave equations from equations (5.26a) and (5.26b).

\[
\overline{V}(z,t) = \overline{T}_v \overline{V}_m(z,t) \quad \overline{I}(z,t) = \overline{T}_i \overline{I}_m(z,t) \tag{5.27}
\]

\[
\frac{d}{dz} \overline{V}_m(z,t) = -\left( \overline{T}_v^{-1} \overline{LT}_i \right) \frac{d}{dt} \overline{I}_m(z,t) \tag{5.28a}
\]

\[
\Rightarrow \frac{d}{dz} \overline{V}_m(z,t) = -\text{diag}(l_1,l_2,\cdots,l_n) \frac{d}{dt} \overline{I}_m(z,t)
\]

\[
\frac{d}{dt} \overline{I}_m(z,t) = -\left( \overline{T}_i^{-1} \overline{CT}_v \right) \frac{d}{dz} \overline{V}_m(z,t) \tag{5.28b}
\]

\[
\Rightarrow \frac{d}{dt} \overline{I}_m(z,t) = -\text{diag}(c_1,c_2,\cdots,c_n) \frac{d}{dz} \overline{V}_m(z,t)
\]
The transformation between \((\mathbf{V}, \mathbf{I})\) and \((\mathbf{V}_m, \mathbf{I}_m)\) through \((\mathbf{T}_V, \mathbf{T}_I)\) in equation (5.27) is implemented in actual circuit using current controlled current sources and voltage controlled voltage sources (Paul 1988). This is shown in Figure 5.6 for one of the termination. It is evident that the transformation network in Figure 5.6 is a linear network. This is only possible when inductance and capacitance matrices are independent of frequency (so that impedance and admittance is linear function of frequency as in \(j\omega L\) and \(j\omega C\)) as in lossless case.

![Diagram](image)

**Figure 5.6** - Implementation of the transformation network for one terminal.

The complete block diagram for implementation of a \(n+1\) parallel multiconductor transmission line system including the linear transformation network of Figure 5.6 is shown in Figure 5.7. Most SPICE circuit simulators support the single uncoupled transmission line model for lossless and lossy condition, therefore creation of the schematic in Figure 5.7 needs not entails incorporation of new component in the solution algorithm as in lossy case where equations (5.25a) and (5.25b) applies. The problem now reduces to finding the appropriate transformation matrices for voltage and current vectors so that the inductance and capacitance matrices can be diagonalized. There are two general lossless conditions for multiconductor transmission line system, first is when the dielectric is homogeneous and second is when the dielectric is not homogeneous.
5.5.2 Homogeneous Dielectric Media

Under the condition where the dielectric on the multiconductor system is homogeneous and the system is lossless, the electromagnetic field propagating within a system of multiconductor transmission line would be of pure TEM mode. The propagating constant $\gamma$ would be imaginary, and is solely dependent on the characteristics of the dielectric surrounding the perfect conductors and the operating frequency. This can be shown by considering the homogeneous Hemholtz wave equation for either $E$ and $H$ field in source-free medium and assuming a solution of the form $E_{o}(x,y)\exp(\pm\gamma z)$ (Collin 1992 or Cheng 1990).

$$\gamma = j\omega \sqrt{\varepsilon \varepsilon_{o} \mu_{o}}$$  \hspace{1cm} (5.29)

There may be more than one TEM modes propagating within the system. However these TEM modes must propagate at the same phase velocity as in
Since the equivalent voltages and currents within the \( n + 1 \) conductors transmission lines system are proportional to the transverse electric and magnetic field, the resultant modes must all propagate with same phase velocity. This implies the matrix \( \mathbf{Z} \) or \( \mathbf{Y} \) from equations (5.6a) and (5.6b) is \( n \)th order degenerate, i.e. they have only one eigenvalue. Considering equation (5.6a) for homogeneous lossless condition:

\[
\left[-\omega^2 \mathbf{L} \mathbf{C} - \gamma^2 \mathbf{U}\right] = 0 \quad \text{which can be written as:}
\]

\[
\left(-\omega^2\right)^{\mathbf{L} \mathbf{C} + \left(\frac{\gamma}{\omega}\right)^2 \mathbf{U}} = 0
\]

The term within the determinant operator in equation (5.30) must be a zero matrix in order to satisfy the equation. Substituting equation (5.29) in equation (5.30):

\[
\mathbf{L} = \left(\sqrt{\epsilon' \epsilon_o \mu_o}\right)^2 \mathbf{C}^{-1}
\]

Thus for the homogeneous lossless condition, the requirement of single propagation velocity results in both the inductance and capacitance matrices be related by equation (5.31). Therefore either inductance or capacitance matrix will be sufficient to fully describe the system.

For realizable network both the inductance and capacitance matrices must be positive definite (Marx 1973). This is illustrated by the fact that since \( \mathbf{L} \) and \( \mathbf{C} \) are real and symmetrical, from matrix theory both matrices must be positive definite (Ayres 1963). Symmetrical positive definite matrices can be diagonalized by orthogonal transformation.

\[
\mathbf{T}^T \mathbf{L} \mathbf{T} = \text{diag}(l_1, l_2, \ldots, l_n)
\]

\[
(5.32a)
\]
where $\overline{T} = \overline{S}$

(5.32b)

Matrix $\overline{T}$ is formed by augmenting the $n$ eigenvectors for $\overline{L}$ into $n \times n$ square matrix row wise or column wise. Using equations (5.31), (5.32a) and (5.32b), the following can be concluded (Paul 1988):

$\overline{T}_{V} = \overline{T}_{I} = \overline{T}$

(5.33a)

$\text{diag}(c_{1}, c_{2}, \cdots, c_{n}) = \frac{1}{\varepsilon_{0}, \varepsilon_{r}, \mu_{0}} \text{diag}(l_{1}, l_{2}, \cdots, l_{n})$

(5.33b)

### 5.5.3 Inhomogeneous Dielectric Media

When the dielectric media is inhomogeneous, the electromagnetic propagating within the multi-conductor transmission line system will no longer be pure TEM. quasi-TEM propagation is assumed and the system will not be nth order degenerate. Nevertheless inductance and capacitance matrices are still real, symmetrical (thus positive definite) and diagonalizable. Carrying out orthogonal transformation on capacitance matrix :

$\overline{A} \overline{C} \overline{A} \overline{D} = \overline{D}$ where $\overline{D} = \text{diag}(d_{1}, d_{2}, \cdots, d_{n})$

(5.34)

where matrix $\overline{A}$ is constructed by augmenting the $n$ eigenvectors of $\overline{C}$. Using the result of equation (5.34), another orthogonal transformation is performed in equation (5.35) since the products within the brackets are real and symmetric.

$\overline{S} \left( \overline{D}^{\dagger} \overline{A} \overline{LAD}^{\dagger} \right) \overline{S} = \text{diag}(l_{1}, l_{2}, \cdots, l_{n})$

(5.35)

where matrix $\overline{S}$ is constructed by augmenting the $n$ eigenvectors of $\overline{D}^{\dagger} \overline{A} \overline{LAD}^{\dagger}$.

By comparison of equation (5.35) with equation (5.28a), it can be verified that:
\[
T_i = AD S \quad (5.36a)
\]
\[
T_v = AD^{-\frac{1}{2}} S \quad (5.36b)
\]
\[
diag(c_1, c_2, \cdots, c_n) = \bar{U} = diag(1,1,\cdots,1) \quad (5.36c)
\]

5.6 Determining the RLCG Matrices for Multiconductor Transmission Line

In determining the \(\bar{R}, \bar{L}, \bar{C}, \bar{G}\) matrices for a multiconductor transmission line system, the following conditions are assumed to be valid:

1. Skin effect loss in the conductors is small with respect to overall power transfer across the transmission line.

2. Dielectric loss (leakage and polarization) of the medium is small with respect to overall power transfer across the transmission line.

Both conditions above are always true for present day high-speed digital PCB as high quality dielectric material and good alloy plating are employed. Changes to the inductance matrix from lossless to lossy condition are due to the additional magnetic flux linkage within the conductor while the present of tangential electric field component on conductor surfaces will alter the elements value in capacitance matrix. Under low loss condition the inductance matrix \(\bar{L}\) and the capacitance matrix \(\bar{C}\) are expected to be slightly different for the case of lossless multiconductor transmission line. Therefore using the perturbation method, it is expected that the resistance and conductance matrices \(\bar{R}\) and \(\bar{G}\) can be estimated from the quasi-TEM lossless electromagnetic field configuration within the transmission line system. The electromagnetic field solution method of Section 2.2 can be applied to determine the \(\bar{L}\) and \(\bar{C}\) matrices. Alternatively for non-
magnetic material ($\mu_r = 1$) the inductance matrix can be determined using equation (5.31):

$$L = \left( \sqrt{\varepsilon \mu} \right)^2 C_o^{-1}$$  \hspace{1cm} (5.37)

with $C_o$ being the capacitance matrix when all dielectric are replaced by free space. However the $R$ cannot be estimated in a straight forward manner as in Section 2.2 or Section 4.3.1 by considering the ratio of power loss over total power transfer. This is due to the presence of modes in the multiconductor transmission line (Frankel 1977). The approach by Harrington and Wei 1984 will be used to estimate the low loss resistance and conductance matrices. The quasi-TEM lossless electromagnetic field can be approximated using FEM (Silvester and Ferrari, 1990) or Method of Moments. Wei et al 1984 provides a detail formulation and algorithm for applying Method of Moments to multiconductor system in layered dielectric environment.

5.6.1 The Conductance Matrix

Considering equation (2.13a) again:

$$C_{ij} = \frac{\delta_0}{V_i V_j} \int \int_{\Omega} \varepsilon E_i E_j d\Omega$$  \hspace{1cm} (2.13a)

If the dielectric loss is present, then the permittivity will become complex as given by equation (4.7):

$$\varepsilon = \varepsilon_r \varepsilon_0 \left( 1 - j \tan \delta \right)$$  \hspace{1cm} (4.7)
Substituting equation (4.7) into equation (2.13a) will result in complex capacitance matrix $C_{\text{complex}}$. Since the elements of the capacitance matrix is proportional to permittivity, multiplying the complex capacitance matrix with $j\omega$, the resulting product becomes:

$$j\omega C_{\text{complex}} = j\omega(1 - j \tan \delta)C = \overline{G} + j\omega \overline{C}$$  \hspace{2cm} (5.38)

Evidently from equation (5.38) it is concluded that:

$$\overline{G} = \omega \tan \delta \overline{C}$$ \hspace{2cm} (5.39)

Typical loss tangent $\tan \delta$ for PCB dielectric material is less than 0.001 (for instance loss tangent for Teflon is around 0.0003 at microwave region). The elements of capacitance matrix are of the order $10^{-12}$ to $10^{-15}$. Therefore the elements of conductance matrix are usually very small at microwave frequency of less than 10GHz and the power loss is negligible. Power loss due to the dielectric is given by:

$$P_{\text{dielectric}} = \frac{1}{2} \overline{V^T G V}$$ \hspace{2cm} (5.40)

The values of voltage and currents used throughout are the peak value.

5.6.2 The Resistance Matrix

From equations (5.14a) and (5.15) it is seen that voltages and currents propagating within the multiconductor transmission line system consist of linear superposition of the modal voltages and currents. Determination of resistance matrix proceeds initially by estimating the skin effect loss of conductors for each mode using perturbation approach. The attenuation constant is then estimated for each mode. Once all modal attenuation constants are determined, the elements of the resistance matrix $\overline{R}$ are calculated. Note that in estimating the conductor loss,
skin effect loss due to ground plane must also be considered if the contribution from ground plane is significant.

Initially equations (5.6a) and (5.13b) are used to determine the modal voltage matrix \( \overline{M}_V \) and modal current matrix \( \overline{M}_I \). Applying each set of modal voltage and current to bias the multiconductor transmission line system, the lossless quasi-TEM electromagnetic field is then approximated either via FEM or Moment of Method. Let superscript \( i \) denotes the \( i \)th mode and subscript \( k \) denotes the \( k \)th conductor. The power loss per unit length due to conductor loss for mode \( i \) is given by:

\[
P^{i}_{\text{cond}} = \frac{1}{2} \sum_{k} R_s \int_{c_k} \left| \mathbf{H}^i_k \right|^2 dl
\]

where \( \mathbf{H}^i_k \) is the lossless mode \( i \) magnetic field on the surface of conductor \( k \) and \( R_s \) is the surface resistance of the conductor. Equation (5.41) is similar to equation (4.13b) except that it is applied to multiconductor. In addition the unperturbed power transfer for mode \( i \) is given by:

\[
P^{i}_{T} = \frac{1}{2} \left( \overline{\mathbf{V}}^{i}_o \right)^{T} \overline{\mathbf{I}}^{i}_o
\]

The subscript “o” in the voltage and current eigenvectors denotes vectors for lossless condition. The modal voltage and current vectors in equation (5.42) are real since \( \overline{LC} \) is real and the eigenvalues (modes) are also real (Marx 1973). Under low loss condition, attenuation due to skin effect loss and dielectric loss can be distinguished as in equation (4.32). The mode \( i \) attenuation constant for skin effect loss is approximately given by the following relation:

\[
\alpha^{i} \approx \frac{P^{i}_{\text{cond}}}{2 P^{i}_{o}}
\]
From equation (5.6a) and assuming $G = 0$:

$$(\alpha' + j\beta')^2 \vec{V}' = \left(\vec{R} + j\omega\vec{L}\right)\left(j\omega\vec{C}\right)\vec{V}'$$  \hspace{1cm} (5.44a)

Imaginary portion of equation (5.44a) is:

$$2\alpha'\beta'\vec{V}' = \omega RCV'$$  \hspace{1cm} (5.44b)

Since losses are small in equation (5.44a):

$$\vec{V}' \approx \vec{V}'_o.$$  \hspace{1cm} (5.45a)

$$\beta' \approx \beta'_o$$  \hspace{1cm} (5.45b)

and  $$I'_o = \frac{\omega}{\beta'} C\vec{V}'_o$$  \hspace{1cm} (5.45c)

Thus using the assumption of equations (5.45a) to (5.45c) in equation (5.44b):

$$2\alpha'\vec{V}'_o \approx \overline{R} I'_o$$  \hspace{1cm} (5.46)

By considering equation (5.46) for each of the $i$th mode, $N^2$ simultaneous equations can be obtained for $N^2$ unknowns in $\overline{R}$. Specifically for the $j$th row of resistance matrix:

$$R_{j1}I^1_{o1} + R_{j2}I^1_{o2} + \cdots + R_{jN}I^1_{oN} = 2\alpha^1V^1_{oj}$$

$$R_{j1}I^2_{o1} + R_{j2}I^2_{o2} + \cdots + R_{jN}I^2_{oN} = 2\alpha^2V^2_{oj}$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$  \hspace{1cm} (5.47)

$$R_{j1}I^N_{o1} + R_{j2}I^N_{o2} + \cdots + R_{jN}I^N_{oN} = 2\alpha^N\overline{V}^N_{oj}$$

where $j = 1, 2, 3, \ldots, N$. 

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The procedure above is used to solve for a single row \( j \) of the resistance matrix. Repeating this for all other rows and the resistance matrix will be fully approximated.

### 5.7 Simulation and Measurement Examples

In this example simulation and measurement for two adjacent striplines are performed. Two identical gold-plated copper striplines are placed 20mils\(^3\) apart from center to center. The lines are designed to have lossless impedance of 50\(\Omega\) in the dielectric FR4 of \(\varepsilon_r = 4.3\). The effective coupling length of the striplines is 9 inches. Capacitance and inductance matrices are estimated using field solution method for lossless condition. Resistance and conductance matrices are estimated using the perturbation method. The per unit length RLCG matrices are assumed to be constant with respect to frequency. Simulations using frequency domain method and impulse response method of Section 5.4 are performed. A measurement is then carried out to examine the correlation between simulation and actual measurement. Finally a time domain circuit simulation assuming lossless transmission model is performed for comparison. The two striplines are connected as shown in Figure 5.8.

\[
C = \begin{bmatrix}
120 \text{pF} & -2.414 \text{pF} \\
-2.414 \text{pF} & 120 \text{pF}
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
300 \text{nH} & 9.78 \text{nH} \\
9.78 \text{nH} & 300 \text{nH}
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
0.08 \Omega & 0.02 \Omega \\
0.02 \Omega & 0.08 \Omega
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
1 \times 10^{-5} & -1 \times 10^{-6} \\
-1 \times 10^{-6} & 1 \times 10^{-5}
\end{bmatrix}
\]
Figure 5.8 - Schematic of two striplines example.

Again it must be stressed for simulation using impulse response method, the sampling point must be twice the number of discretization (equation (4.64)) for numerical integration using rectangular rule as in Section 4.5.1. Comparison for the results is shown in Figure 5.10A for the active line and Figure 5.10B for the passive line. Good correlation between frequency domain method and impulse response method is obtained. The shape of the stimulus in the active line is shown in Figure 5.9. Rise time and fall time of the stimulus is approximately 100ps while amplitude is 250mV.

Figure 5.9 - Source waveform on the active line.

---

3 1 mils is equivalent to one thousandth of an inch
A measurement on the physical board is performed for the purpose of comparison. The experimental setup for measurement is as depicted in Figure 5.11. Results from measurement are in Figures 5.12A and 5.12B for active line and passive line respectively. The waveforms in Figures 5.12A and 5.12B have been expanded on a smaller time scale, therefore only a portion of the crosstalk or
coupled signal due to positive-going cycle of the stimulus is seen as compared to Figures 5.10A and 5.10B.

- Setup for measurement.

**Figure 5.12A** - Measurement on active line.

**Figure 5.12B** - Measurement on passive line.
The coupled voltage are known as forward crosstalk at the far end of the passive line from the source and backward crosstalk and the near end (Feller et al 1965, Catt 1967). From Figures 5.10B, the forward crosstalk level from simulation is -9mV while measurement recorded -3mV. The measured backward crosstalk level is +2mV to +3mV, while simulation provides +3mV to +4.5mV. Possible reasons for the discrepancy observed are the assumption that RLCG matrices are constant with respect to frequency and also the error in determining the elements of the matrices using field solution method. Another important reason is simulation assumes ideal interconnection between the source, resistors and the transmission lines, whereas the experimental PCB used for the measurement employs SMA type coaxial to PCB adapters which would cause substantial degradation of signal, hence the lower levels.

Finally a lossless time domain circuit simulation is performed for the two line system. The equivalent circuit model is constructed based on the proposal of Section 5.5. Assuming homogeneous dielectric medium for the striplines, the transformation matrix is given by:

\[
\begin{bmatrix}
0.7071 & 0.7071 \\
-0.7071 & 0.7071
\end{bmatrix}
\]

And the transformed per unit length inductance and capacitance matrices are:

\[
\begin{bmatrix}
2.9022 \times 10^{-7} & 0 \\
0 & 3.0978 \times 10^{-7}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.2241 \times 10^{-10} & 0 \\
0 & 1.1759 \times 10^{-10}
\end{bmatrix}
\]

A listing of the SPICE netlist based on Paul 1988 is shown in Listing 5.1 and the simulated results are in Figure 5.13A and Figure 5.13B.
Figure 5.13A - Time domain simulation waveforms for passive line.

Figure 5.13B - Time domain simulation waveforms for active line.

Despite the fact that differences in signal levels are observed between frequency domain simulation, measurement and lossless equivalent circuit simulation, all the coupled waveforms exhibit the same characteristic shape. Catt 1967 and Feller et al 1965 provide both quantitative and qualitative explanation of the shapes and levels of forward and backward crosstalks.

* MTL implementation of stripline
Vs 1 0 PULSE (0 0.5 0 100p 100p 1u 2u )
Rs 1 2 50
V1 2 3 0
RL 9 0 50
V3 9 8 0
RFE 10 0 50
V4 10 11 0
RNE 17 0 50
V2 17 16 0
E1 3 4 5 0 0.7071
E2 4 0 14 0 -0.7071
E3 8 7 6 0 0.7071
Listing 5.1 - Coupled stripline model.
Figure 5.14 - Schematic of the dual transmission line model.