CHAPTER 4 Modeling of Single Lossy Transmission Line

4.1 Introduction

With the advent of deep submicron technology in semiconductor industry where operating frequency of device doubled every five years, dimension of the trace within the metal layer of the die decreases, hence increasing the resistive loss of the metal conductor. Improvement in component placement technology results in printed circuit board with higher component density, forcing the use of finer copper trace width in PCB. Higher packaging density also results in finer connection between die and pins, thus producing higher electrical loss. Effect of lossy transmission line becomes conspicuous when length of line exceeds the shortest wavelength propagating along it. Three main loss mechanisms are metallic loss within the conductor, leakage and polarization loss in the dielectric material and radiation loss. Radiation losses occur at imperfections and discontinuities in transmission lines, it will not be considered here. In quasi-TEM and pure TEM mode of propagation, these mechanisms can be represented by a distributed series resistor and a distributed shunt conductance. Note that in this chapter, the main assumptions are (i) all conductor and dielectric material are non-magnetic. (ii) losses attributed to dielectric and conductor is very small compared to overall power transfer so that perturbation method may be used. Balanis 1989 provides a good overview of loss mechanism in dielectric. While Collin 1992 derives the lossy transmission line parameters through perturbation method with the assumption that the effect of dielectric and conductor loss is independent. Here the concepts introduced by these two authors will be adopted.

Specifically the writer reformulates the problems to derive analytical expressions for phase constant $\beta$ and attenuation constant $\alpha$ using field analysis and circuit theory approaches. It is shown that both approaches yield similar expressions under the assumption of low loss transmission line. Once $\alpha$ and $\beta$ are known, standard equivalent mathematical model can be constructed for the low loss transmission line (Paul 1988, Cheng 1990). Simulations are then carried out
for lossy transmission line using both Fourier Transform approach and Time Domain Convolution approach. For the case of Time Domain Convolution approach, it is shown in Section 4.6.1 that for accurate results, the discretization of time axis for numerical integration must be twice the highest harmonics of the impulse response considered, otherwise there will be a residue d.c. error in the signal.

4.2 AC Dielectric Permittivity and Loss Tangent
Normally all dielectric materials are lossy. The loss is due to dielectric leakage and polarization loss. Four important mechanisms contribute to polarization. These are:

- Dipole or Orientation Polarization
- Ionic or Molecular Polarization
- Electronic Polarization
- Interfacial Polarization

Hence under a.c. or transient condition the effective a.c. permittivity is a complex number. This can be shown by adopting the phasor notation using \( E_0 e^{j\omega t} \) as the complex field and applying mechanical equivalent for a simple electron-nucleus system as shown in Figure 4.1.

![Figure 4.1 - Analogy between simple molecular model and mechanical system.](image)

From the Figure 4.1 we could form a second order differential equation:

\[
\frac{d^2s}{dt^2} + \frac{d}{m \, dt} + \frac{k}{m} \, s = \frac{Q}{m} E_0 e^{j\omega t} \quad (4.1)
\]
The solution contains two portions, the transient portion which is given by the auxiliary solution and the steady state which is given by the particular solution. Here we are interested in the steady state solution. Roots for the auxiliary solution of equation (4.1) is given by:

\[
\text{roots}_{1,2} = \frac{1}{2} \left[ -\left(\frac{d}{m}\right) \pm \sqrt{\left(\frac{d}{m}\right)^2 - 4\left(\frac{k}{m}\right)} \right] = \frac{1}{2m} \left[ -d \pm \sqrt{d^2 - 4km} \right] \quad (4.2)
\]

\[
s(t) = e^{2m} \left( C_1 \exp \left\{ \frac{\sqrt{d^2 - 4km}}{2m} t \right\} + C_2 \exp \left\{ -\frac{\sqrt{d^2 - 4km}}{2m} t \right\} \right) + S_o e^{j \omega t} = s(t) + s_p(t)
\]

(4.3)

Only the particular solution, which gives the steady state response, is of interest. Substituting the particular solution into equation (4.1) and solving for \(S_o\):

\[
s_p(t) = \left( \frac{Q}{m} \right) E_o \left( \frac{e^{j \omega t}}{\omega_o - \omega^2} + j \omega \left( \frac{d}{m} \right) \right) \quad \text{where} \quad \omega_o = \sqrt{\frac{k}{m}} \quad (4.4)
\]

Polarization vector is the polarization per unit volume:

\[
P = NQs_p = \chi E = \left[ \frac{NQ^2}{\omega_o^2 - \omega^2 + j \omega \left( \frac{d}{m} \right)} \right] E = \left[ \chi_{\text{real}} - j \chi_{\text{im}} \right] E \quad (4.5)
\]

where \(Q\) is the polarization charge and \(N\) is the dipole per unit volume.

From \(\varepsilon = \varepsilon_o \varepsilon_r = \varepsilon_o \left( 1 + \frac{P}{E \varepsilon_o} \right)\), we have:

\[
\varepsilon_r = (1 + \chi) = (1 + \chi_{\text{real}}) - j \chi_{\text{im}} = \varepsilon' - j \varepsilon'' \quad (4.6a)
\]

With \(\varepsilon_r' = 1 + \frac{(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2 + (\omega \left( \frac{d}{m} \right))^2)} \left( \frac{NQ^2}{\varepsilon_o m} \right)\)

(4.6b)

the a.c. dielectric constant or real permittivity and
\[
\varepsilon_r'' = \frac{NQ^2}{\varepsilon_0 m} \left( \frac{\omega \frac{d}{m}}{(\omega_0^2 - \omega^2) + (\omega m)^2} \right)
\]

(4.6c)

the a.c. loss factor or imaginary permittivity. Consider the time harmonics Maxwell equation for magnetic intensity:

\[
\nabla \times \vec{H} = \vec{J}_{total} = \vec{J}_{imposed} + \sigma \vec{E} + j \omega \varepsilon_r (\varepsilon_r' - j \varepsilon_r '') \vec{E}
\]

\[
\Rightarrow \nabla \times \vec{H} = \vec{J}_{imposed} + j \omega \varepsilon_r \varepsilon_r' \left[ 1 - j \left( \frac{\varepsilon_r''}{\varepsilon_r'} \right) + \frac{\sigma}{\omega \varepsilon_r} \right] \vec{E} = \vec{J}_{imposed} + j \omega \varepsilon_r \varepsilon_r' (1 - j \tan \delta) \vec{E}
\]

(4.6d)

Thus the effective dielectric constant incorporating polarization loss and dielectric leakage is:

\[
\varepsilon_{effective} = \varepsilon_r' \left[ 1 - j \left( \frac{\varepsilon_r''}{\varepsilon_r'} \right) + \frac{\sigma}{\omega \varepsilon_r} \right] = \varepsilon_r' - j \varepsilon_r' \tan \delta
\]

(4.7)

The term \( \tan \delta \) is referred to as the loss tangent. The loss tangent is composed of two components, one is the leakage current, which exists in the dielectric, and second is the inherent loss due to polarization within the dielectric molecular structure.

### 4.3 RLCG Circuit Parameters of Lossy Transmission Line

A lossy transmission line with its dominant mode of propagation being TEM or quasi-TEM can be represented by distributed parameters of the form shown in Figure 4.2, Dworsky 1979.

\[\text{Figure 4.2} - \text{Distributed representation of lossy transmission line, assuming TEM and quasi-TEM propagation.}\]
Assuming a lossy transmission line parallel to the z-axis. The field solutions for E and H field satisfying Helmholtz wave equation and propagating in positive z direction are assumed to be:

\[
E = E^0(x, y)e^{-(\alpha + j\beta)z} \quad H = H^0(x, y)e^{-(\alpha + j\beta)z}
\]

where \(\alpha\) is the attenuation constant and \(\beta\) is the phase constant. The voltage between any two conductors in the transmission line will be proportional to the transverse components (x and y component) of the electric field while the current along the line at any time is proportional to the transverse component of the magnetic field. Therefore we can analyze the transmission line in terms of its equivalent voltage and current in circuit theory as in Figure 4.2 or in terms of its actual electric and magnetic field. Let \(P\) be the power propagating along the transmission line and is given by the integral of the Poynting vector over the line’s cross section. After the electromagnetic wave travels a distance of \(l\), the propagated power will diminish by a factor of \(\exp(-2\alpha l)\) because power is proportional to the square of voltage. This reduction in power is equivalent to power loss.

\[
-\frac{d}{dt} P = P_{\text{loss}} = 2\alpha P^2 e^{-2\alpha l} = 2\alpha P
\]

\[
\alpha = \frac{P_{\text{loss}}}{2P}
\]

Most of the time the dielectric and conductor loss for a transmission line is very small. Isolation resistance for commercial dielectric can be in excess of 10MΩ while loss tangent for good dielectric is smaller than 0.01 up to microwave frequencies of 5GHz. Furthermore the conductor used in waveguide and transmission lines are very good conductors, for example conductivity of copper is 5.8x10⁷ Siemens or more. In this case the tangential electric field in the conductor is very small and the actual electromagnetic field is almost similar as the field for lossless condition. By determining the electric and magnetic field surrounding the conductors of a transmission line, the distributed parameters R, L, C and G can then be calculated. Solution for the E and H field can be achieved through electromagnetic field solver software using methods such as Method of Moments (MoM) or Finite Elements Method (FEM). Often the field solution for the lossy
case is very difficult to find, in which case the perturbation method is extremely useful and simple to carry out. This method is based on the assumption that the introduction of small loss does not substantially disturb the field from its lossless condition. The known field distribution for the lossless case is then used to evaluate the loss parameters in the system, and from this the attenuation constant and phase constant due to non-ideal transmission line can be estimated.

4.3.1 Analysis for Metallic Loss

A practical conductor exhibits a surface impedance given by:

\[
Z_s = \frac{E_{\text{surface}}}{H_{\text{surface}}} = \frac{1 + j}{\sigma \delta_s} = R_s + jX_s \quad \text{where} \quad \delta_s = \text{skin depth} = \sqrt{\frac{2}{\omega \mu \sigma}} \quad (4.10)
\]

A tangential component of the electric field at the surface of the conductor must be present, given by:

\[
E_{\text{tangential}} = Z_s J_s \quad (4.11)
\]

The surface current density is given by:

\[
J_s = n \times H \quad (4.12)
\]

where \( n \) is a unit vector pointing outwards from the conductor surface, Figure 4.3.

Time averaged power loss per unit length for the surface impedance of the line is:

\[
P_{\text{loss}} = \frac{1}{2} \Re Z_s \int_{c_1}^{c_2} J_s \cdot J_s^* \, dl \quad (4.13a)
\]

![Figure 4.3 - Cross section of a two conductor transmission line.](image)

Inserting equations (4.10) and (4.12) into equation (4.13a) and after simplification yields:
The transmission line loss from finite conductivity may be accounted for by a series resistance \( R \) per unit length provided \( R \) is chosen so that:

\[
\frac{1}{2} R I_o^2 = \frac{1}{2} R_s \int |\mathbf{H}|^2 \, dl
\]

(4.14)

The right side of equation (4.14) gives the total power loss per unit length arising from the high frequency resistance of the conductors. In terms of this quantity distributed resistance can be defined as:

\[
R_s = \frac{R_s}{2} \int |\mathbf{H}|^2 \, dl = \frac{\int |\mathbf{H}|^2 \, dl}{\int |\mathbf{H}| \, dl}^2
\]

(4.15)

where \( I_o \) is the peak amplitude for the line current. A further effect of the finite conductivity is to increase the series inductance of the line by a small amount because of the penetration of the magnetic field into the conductor. This skin effect inductance \( L_s \) is readily evaluated on an energy basis. Without going into further analysis (Collin 1992, chapter 4) the relation between skin effect inductance and skin effect resistance is:

\[
\omega L_s = R_s
\]

(4.16)

### 4.3.2 Analysis for Dielectric Loss

If the dielectric is lossy, the equivalent relative permittivity is complex and is given by equation (4.7):

\[
\varepsilon = \varepsilon' (1 - j \tan \delta)
\]

(4.17a)

The total shunt current leaving a conductor consist of displacement current \( I_D \) and conduction current \( I_C \). From \( \nabla \times \mathbf{H} = \mathbf{J}_{\text{imposed}} + \mathbf{J}_{\text{displacement}} = \mathbf{J}_{\text{imposed}} + j \omega \varepsilon_o \varepsilon \mathbf{E} \) and

Performing line integration on either conductor:

\[
I = I_C + I_D = j \omega \varepsilon_o \int_{C_2} \mathbf{E} \cdot d\mathbf{l} \quad \text{which upon expanding}
\]
\[ I_D + I_C = j \omega \varepsilon_\varepsilon' \oint_{c_2} \mathbf{E} \cdot d\mathbf{l} + \omega \varepsilon_\varepsilon' \tan \delta \oint_{c_2} \mathbf{E} \cdot d\mathbf{l} \]  \hspace{1cm} (4.18)

The total shunt admittance is given by:

\[ Y = j \omega C + G = \frac{I_D + I_C}{V_o} \]  \hspace{1cm} (4.19)

where \( V_o \) is the amplitude of line voltage. Therefore it is seen that:

\[ G = \frac{I_C}{V_o} = \frac{I_C}{I_D} \frac{I_D}{V_o} = \omega (\tan \delta) C \]  \hspace{1cm} (4.20)

### 4.3.3 Summary of RLCG Parameters in Field Analysis

The distributed L and C parameters for a transmission line can be determined from an evaluation of the stored electric and magnetic field energy. Energy stored in the magnetic field is accounted for by the series inductance L and energy stored in electric field is accounted by the distributed shunt capacitance C. Power loss in the conductors is taken into account by series resistance R and power loss in dielectric is taken into account by shunt conductance G. Note that in the case of using perturbation concept, the \( \mathbf{E} \) and \( \mathbf{H} \) fields are the fields computed for lossless condition.

\[ L = \frac{\mu_o}{I_o I_o^*} \iint_S \mathbf{H} \cdot \mathbf{H}^* dS \]  \hspace{1cm} (4.21a)

\[ C = \frac{\varepsilon_o \varepsilon'}{V_o V_o^*} \iint_S \mathbf{E} \cdot \mathbf{E}^* dS \]  \hspace{1cm} (4.21b)

\[ R = \frac{R_i}{I_o I_o^*} \oint_{c_1+c_2} \mathbf{H} \cdot \mathbf{H}^* d\mathbf{l} \]  \hspace{1cm} (4.21c)

\[ G = \frac{\omega \varepsilon_\varepsilon' \varepsilon'''}{V_o V_o^*} \iint_S \mathbf{E} \cdot \mathbf{E}^* dS \]  \hspace{1cm} (4.21d)

Equation (4.21a) to (4.21d) in this section is based on energy consideration and are supposed to be valid for all propagation modes (Collin 1992).

### 4.4 The Attenuation Factor and Phase Factor of a Lossy Transmission Line
It is observed that both dielectric and conductor losses affect each other. Hence the expression for total attenuation factor $\alpha$ is expected to consist of coupled $R$ and $G$. However under conditions such as:

- High operating frequency (lower than the transverse resonance frequency of the transmission line cross section)
- Low dielectric loss
- Low conductor loss

A simplified attenuation constant $\alpha$ and phase constant $\beta$ can be obtained. The derivation below will attempt to prove this using both field analysis approach and circuit theory analysis approach. The total power propagating along the line is given as real part of Poynting vector integral:

$$P = \left| \text{Re} \frac{1}{2} \iint_S (E \times H^\ast) \bullet dS \right| = \frac{1}{2} \text{Re}(Z) \iint_S H \bullet H^\ast dS$$  \hspace{1cm} (4.22a)$$

with $Z = \sqrt{\frac{\mu}{\varepsilon}}$ the intrinsic impedance in the dielectric, using $1 - j\tan\delta = (\sec\delta) e^{\frac{j\delta}{2}}$, $\text{Re}(Z)$ can be expressed as:

$$\text{Re}(Z) = \text{Re} \left[ \frac{\mu_o}{\varepsilon_o \varepsilon'(1 - j\tan\delta)} \right] = \frac{1}{\sqrt{\varepsilon'}} \left( \frac{\cos\frac{\delta}{2}}{\sqrt{\sec\delta}} \right) \frac{Z_o}{\text{Re}(1/Y)} \hspace{1cm} (4.22b)$$

Where:

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \hspace{1cm} (4.22c)$$

Therefore the from equations (4.9) and (4.13b), equations (4.22a) and (4.22b), the conductor attenuation factor can be written as (observing that $\sqrt{\cos\delta} \cos\frac{\delta}{2} = \sqrt{2\cos\frac{\delta}{2} - 1} \cos\frac{\delta}{2}$):

$$\alpha_{\text{conductor}} = \frac{P_{\text{loss}}}{2P} = \left( \sqrt{\text{sec}\delta} \right) \frac{\sqrt{\varepsilon'} R_s}{\cos\frac{\delta}{2}} \iint_S \frac{H \cdot H^\ast dl}{2Z_o} \iint_S H \cdot H^\ast dS$$  \hspace{1cm} (4.23)$$
Using $Z_c = \sqrt{\frac{1}{\varepsilon}} = 1/Y_c$ and equation (4.21a), the conductor loss attenuation factor can be written as (using $\sqrt{LC} = \sqrt{\varepsilon \varepsilon_0 \mu_0}$):

$$\alpha_{\text{conductor}} = \left| \frac{\sec \delta}{\cos \frac{\delta}{2}} \right| \frac{R_c \mu_0}{c_1 + c_2} \int \mathbf{H} \cdot \mathbf{H'} dl \left| \frac{\sec \delta}{\cos \frac{\delta}{2}} \right| \frac{R_c \mu_0 Y_c}{2ZL} \sqrt{\frac{L}{C}} \quad (4.24)$$

$$\alpha_{\text{conductor}} = \left| \frac{\sec \delta}{\cos \frac{\delta}{2}} \right| \frac{R_c Y_c}{2} \quad (4.24)$$

An important observation is that the effect of dielectric loss contributes to the conductor attenuation factor through the term ($\sqrt{\sec \delta / \cos \frac{\delta}{2}}$). For low loss dielectric:

$$\alpha_{\text{conductor}} \equiv \frac{R_c Y_c}{2} \quad \text{for low loss dielectric} \quad (4.25a)$$

The effective leakage current in the dielectric is:

$$\mathbf{J} = (\varepsilon' \omega \tan \delta) \mathbf{E} \quad (4.26)$$

where the bracketed term in equation (4.26) is the equivalent conductivity. Referring to Figure 4.3 where $S$ is a closed surface encapsulating either one of the conductor, the power loss due to dielectric is given as (from integral form of $P=0.5I^2R$):

$$P_{\text{loss}} = \frac{1}{2(\varepsilon' \omega \tan \delta)} \iint_S \mathbf{J} \cdot \mathbf{J'} ds = \left( \frac{\varepsilon' \omega \tan \delta}{2} \right) \iint_S \mathbf{E} \cdot \mathbf{E'} ds \quad (4.27)$$

and the total power propagating along the line is given as real part of Poynting vector integral:

$$P = \left| \frac{\text{Re}}{2} \iint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \right| = \frac{1}{2} \text{Re}(Y) \iint_S \mathbf{E} \cdot \mathbf{E'} dS \quad (4.28)$$

Again from equations (4.9), (4.22b), (4.27) and (4.28):

$$\alpha_{\text{dielectric}} = \frac{\varepsilon' \omega \tan \delta}{2Y} \left( \sqrt{\sec \delta / \cos \frac{\delta}{2}} \right) \quad (4.29)$$

This can be further simplified using equation (4.20) and $Z_c = \sqrt{\frac{1}{\varepsilon}}$.
\[ \alpha_{\text{dielectric}} = \left( \sqrt{\sec \delta \cos \frac{\delta}{2}} \right) \frac{\varepsilon_0 \sqrt{\varepsilon} G}{2Y_GC} Z_i \sqrt{\frac{C}{L}} = \left( \sqrt{\sec \delta \cos \frac{\delta}{2}} \right) \frac{GZ_i}{2} \quad (4.30) \]

\[ \alpha_{\text{dielectric}} \equiv \frac{GZ_i}{2} \quad \text{for low loss dielectric} \quad (4.31) \]

When both dielectric and conductor loss are present, the attenuation constant can be defined as:

\[ \alpha = \frac{P_{\text{loss(dielectric)}} + P_{\text{loss(conductor)}}}{2P} \geq \frac{GZ_i}{2} + \frac{RY_G}{2} \quad (4.32) \]

The total attenuation factor is merely addition of \( \alpha_{\text{conductor}} \) and \( \alpha_{\text{dielectric}} \). It is thus seen that the attenuation is dispersive as \( G, R, L \) and \( C \) are dependent on frequency. The above equations are only valid for a homogenous medium. For inhomogeneous medium such as microstrip line, the problem region has to be divided into region of homogenous dielectric characteristic and the relations applied separately to each region.

An alternative approach to field analysis is through consideration of the transfer function of Figure 4.2 is given below. The transfer function for lossy transmission line shown in Figure 4.2 is given by:

\[ H(\omega) = \exp \left[ \left( (R + j\omega L)(G + j\omega C) \right)^{\frac{1}{2}} \right] = e^{\alpha z} = e^{\alpha e^{j\gamma z}} \quad (4.33) \]

From equation (4.33) it is seen that:

\[ \alpha = \text{Re}\left[ (R + j\omega L)(G + j\omega C) \right]^{\frac{1}{2}} \]

\[ \beta = \text{Im}\left[ (R + j\omega L)(G + j\omega C) \right]^{\frac{1}{2}} \quad (4.34) \]

For lossy transmission line, the transfer function will be dispersive as propagation constant \( \beta \) is a function of frequency. The attenuation factor \( \alpha \) is also dependent of frequency. For low loss condition and high operating frequency, the relation between attenuation factor and \( R, L, C \) and \( G \) becomes a simple expression. \( \gamma \) in equation (4.33) can be written as:
\[
\gamma = \left[ (R^2 + \omega^2 L^2) \left( G^2 + \omega^2 C^2 \right) \right]^{\frac{1}{2}} \exp \left[ j \tan^{-1} \left( \frac{\omega(LG + RC)}{RG - \omega^2 LC} \right) \right] (4.35)
\]

Observe that in equation (4.35) the argument is usually a very small value because of \( \omega^2 \) term at the denominator. The equation can be approximated without significant loss of accuracy as (using \( 0.5 \tan^{-1} \theta \equiv \tan^{-1} (0.5\theta) \) for \( \theta \rightarrow 0 \)):

\[
\gamma \approx \left[ (R^2 + \omega^2 L^2) \left( G^2 + \omega^2 C^2 \right) \right]^{\frac{1}{2}} \exp \left[ j \tan^{-1} \left( \frac{\omega(LG + RC)}{RG - \omega^2 LC} \right) \right] (4.36)
\]

The term \( (A)^4 \) has roots \( A, -A, jA \) and \( -jA \). In the case of equation (4.35):

\[
\left[ (R^2 + \omega^2 L^2) \left( G^2 + \omega^2 C^2 \right) \right]^{\frac{1}{2}} = -j \left[ (R^2 + \omega^2 L^2) \left( G^2 + \omega^2 C^2 \right) \right]^{\frac{1}{2}} (4.37)
\]

Equation (4.37) will be considered as the roots as this will fulfill the limiting condition when the lossy transmission line approaches lossless condition. Upon using equation (4.37):

\[
\alpha = \text{Re}(\gamma) \equiv \left[ (R^2 + \omega^2 L^2) \left( G^2 + \omega^2 C^2 \right) \right]^{\frac{1}{2}} \sqrt{\frac{\omega(LG + RC)}{\omega(LG + RC)^2 + 4(RG - \omega^2 LC)^2}}
\]

(4.38a)

Upon simplification of equation (4.38a):

\[
\alpha \approx \left[ (RG - \omega^2 LC)^2 \right]^{\frac{1}{2}} \left[ 1 + \frac{\omega^2 (LG + RC)^2}{(RG - \omega^2 LC)^2} \right]^{\frac{1}{2}} \omega(LG + RC) \sqrt{\omega^2 (LG + RC)^2 + 4(RG - \omega^2 LC)^2}
\]

(4.38b)

For low loss (\( R \rightarrow 0 \), \( G \rightarrow 0 \)) and high operating frequency where \( \omega^2 LC \gg RG \), using binomial expansion:

\[
\alpha \approx \left[ \sqrt{RG - \omega^2 LC} \right] \left[ 1 + \frac{\omega^2 (LG + RC)^2}{4(RG - \omega^2 LC)^2} \right] \omega(LG + RC) \sqrt{\omega^2 (LG + RC)^2 + 4(RG - \omega^2 LC)^2}
\]

(4.39a)

Note that only the first term of the nominator is considered due to the 2nd power operator in equation (4.38b).
\[ \alpha = \sqrt{\omega^2 LC - RG} \left[ \sqrt{\omega^2 (LG + RC)^2 + 4(RG - \omega^2 LC)^2} \right] \frac{\omega(LG + RC)}{4(RG - \omega^2 LC)^2} \]

\[ \alpha|_{R \to 0, G \to 0} = \frac{\omega \sqrt{LC} \cdot \omega(LG + RC) \cdot \sqrt{4\omega^2 L^2 C^2}}{4(\omega^4 L^2 C^2)} \] (4.39b)

and upon further simplification:

\[ \alpha = -\frac{R}{2 \sqrt{L}} + \frac{G}{2 \sqrt{C}} = \alpha_{\text{dielectric}} + \alpha_{\text{conductor}} \] (4.40a)

or,

\[ \alpha = \frac{1}{2} Z_c (Y_c^2 R + G) \] (4.40b)

where the terms in the brackets are referred to as total loss parameters. Similar operations will be carried out to derive the phase constant \( \beta \) in terms of RLCG parameters.

\[ \beta \equiv \left\{ \left( R^2 + \omega^2 L^2 \right) \left( G^2 + \omega^2 C^2 \right) \right\}^{\frac{1}{2}} \frac{2(RG - \omega^2 LC)}{\sqrt{\omega(LG + RC)^2 + 4(RG - \omega^2 LC)^2}} \] (4.41)

Again using Binomial expansion and the conditions as in (4.38a) to (4.40a):

\[ \beta \equiv \sqrt{\omega^2 LC - RG} \left[ \sqrt{\omega^2 (LG + RC)^2 + 4(RG - \omega^2 LC)^2} \right] \frac{2(RG - \omega^2 LC)}{4(RG - \omega^2 LC)^2} \]

Which can be simplified to:

\[ \beta = \frac{\omega \sqrt{LC}}{2} \] (4.42)

Equations (4.40) and (4.42) give the attenuation constant and phase constant for a lossy transmission line with low loss and operating at sufficiently high frequency (which is usually the case for \( f > 100 \text{Mhz} \) \( \omega^2 LC \gg RG \) but below the transverse resonance limit of the structure.

### 4.5 Circuit Elements Representation for Lossy Transmission Line
Since the general lossy transmission line is dispersive, it is logical to begin with a frequency domain representation of the transmission line. This is a generalization of Brahmin’s form (Paul 1988). Solution for voltage and current waves within the general lossy transmission line in z -direction is given by:

\[
V(z, \omega) = \left[ V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \right] e^{j \omega z} \tag{4.43a}
\]

\[
I(z, \omega) = \frac{1}{Z_c} \left[ V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z} \right] e^{j \omega z} \tag{4.43b}
\]

where \( \gamma = \alpha + j \beta \) and \( Z_c = \sqrt{\frac{R + j \omega L}{G + j \omega C}} \) \( (4.43c) \)

For convenience, the term \( \omega \) will be dropped from all the voltage and current variables as it had been explicitly stated that we are dealing with frequency domain. Dropping the \( e^{j \omega t} \) term and considering a transmission line of length \( L \):

\[
V(0) = V_o^+ + V_o^- \tag{4.44a}
\]

\[
Z_c I(0) = V_o^+ - V_o^- \tag{4.44b}
\]

\[
V(L) = V_o^+ e^{-j \beta L} + V_o^- e^{j \beta L} \tag{4.44c}
\]

\[
Z_c I(L) = V_o^+ e^{-j \beta L} - V_o^- e^{j \beta L} \tag{4.44d}
\]

Subtracting and adding equations (4.44a) and (4.44b) :

\[
V(0) + Z_c I(0) = 2V_o^+ \tag{4.45a}
\]

\[
V(0) - Z_c I(0) = 2V_o^- \tag{4.45b}
\]

Again subtracting and adding equations (4.44c) and (4.44d) :

\[
V(L) + Z_c I(L) = 2V_o^+ e^{-j \beta L} \tag{4.46a}
\]

\[
V(L) - Z_c I(L) = 2V_o^- e^{j \beta L} \tag{4.46b}
\]

To eliminate the coefficient terms \( V_o^+ \) and \( V_o^- \), substituting equation (4.46b) into equation (4.45b) :

\[
V(0) = Z_c I(0) + \left\{ [V(L) - Z_c I(L)] e^{-j \beta L} \right\} \tag{4.47a}
\]

Similarly substituting equation (4.45a) into equation (4.46a) :

\[
V(L) = -Z_c I(0) + \left\{ [V(0) + Z_c I(0)] e^{-j \beta L} \right\} \tag{4.47b}
\]
Equations (4.47a) and (4.47b) can be implemented using linear circuit elements such as controlled sources and phase delays. The model shown in Figure 4.4 is an example of the implementation and is only valid for sinusoidal wave function at one frequency \( \omega \). The parameters can be varied for difference frequency and the time domain response can be constructed through Discrete Fourier Transform (DFT).

**Figure 4.4** - Implementation of a lossy transmission line for sinusoidal function or in frequency domain.

Equations (4.47a) and (4.47b) can be written in matrix form:

\[
\begin{bmatrix}
V(0) \\
V(L)
\end{bmatrix} = Z_c \begin{bmatrix}
1 & -e^{-j\alpha L} \\
e^{-j\alpha L} & -1
\end{bmatrix} \begin{bmatrix}
I(0) \\
I(L)
\end{bmatrix} + \begin{bmatrix}
0 & e^{-j\alpha L} \\
e^{-j\alpha L} & 0
\end{bmatrix} \begin{bmatrix}
V(0) \\
V(L)
\end{bmatrix}
\]  
(4.48)

After some manipulation:

\[
\begin{bmatrix}
I(0) \\
I(L)
\end{bmatrix} = Z_c \left( e^{-j\beta L} - 1 \right) \begin{bmatrix}
-(e^{-2j\beta L} + 1) & 2e^{-2j\beta L} \\
-2e^{-2j\beta L} & (e^{-2j\beta L} + 1)
\end{bmatrix} \begin{bmatrix}
V(0) \\
V(L)
\end{bmatrix}
\]  
(4.49)

Bear in mind that equation (4.47) to (4.49) represent network equation for lossy transmission line in frequency domain. For general lossy transmission line with Thevenin or Norton equivalent representation at both ends, equation (4.49) will be used in conjunction with the network equation for the termination at both ends to derive the steady state sinusoidal response. We must multiply these equations with \( e^{j\omega t} \) and taking the real part to obtain the time domain steady state sinusoidal response.

### 4.5.1 Simulation of Lossy Transmission Line
There are two approaches to simulation of a lossy transmission line. The first approach applies Fourier analysis or Fourier transform to determine the spectral distribution of a certain input signal and finding the response for specific discrete frequency components using equation (4.49). A numerical inverse Fourier transform is performed for the discrete frequency response to find the approximate time domain response. This method is only applicable if the termination networks at both ends of the transmission line are linear time invariant system. Consider for example a linear termination network depicted by:

\[ v(t) = f[i(t)] \] (4.50)

Assuming the Fourier Transform for \( i(t) \) and \( v(t) \) exist,

\[ v(t) = f \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{i\omega t} d\omega \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) f(e^{i\omega}) d\omega \] (4.51)

The response of \( f(e^{i\omega}) \) is also sinusoidal, \( Ae^{i\theta} e^{j\alpha} \). This would implies \( v(t) \) of equation (4.51) is simply the summation of infinitesimal responses of various frequency components. Similar argument holds if we choose to use periodic signal and Fourier series instead. For a non-linear system equation (4.51) does not hold since the sinusoidal forcing function does not necessarily produces a sinusoidal response. Only time domain simulation can be performed for a transmission line with non-linear termination. A method proposed by Djordjevic 1987 derive equivalent time domain model for transmission line through impulse response. The low loss transmission line model is essentially linear therefore one can represent the transmission line in time domain using the following relation:

\[ V(\omega) = H(\omega)I(\omega) \leftrightarrow v(t) = \int_{-\infty}^{\infty} h(t-s)i(s)ds = h(t)*i(t) \] (4.52)

The operator \( * \) denotes convolution and \( h(t) \) is the time domain impulse response. Assuming now terminal \( z = L \) is short-circuited and a voltage is applied at terminal \( z = 0 \). The frequency domain current at both terminals is known as \( I(k)_0 \), where \( k = L \) or 0. One can obtain the expression for currents using equation (4.49) by letting \( V(L) = 0 \). All these currents can be represented in the form:

\[ I(k)_0 = Y_{k0}V(0) \quad k = 0 \text{ or } L. \] (4.53)
When \( V(0) \) is unity, \( I(k) \) represents the impulse response of the lossy transmission line system at terminal \( z = k \) and in time domain it is written as:

\[
i_g(k,t)_0 = F^{-1}\{Y_{k0}\}
\]  

(4.54)

The subscript ‘\( g \)’ simply stands for Green’s function. For arbitrary voltages \( V(0) \), the product operator in equation (4.53) becomes a convolution operator in time domain. Assuming no initial energy storage in the system:

\[
i(k,t)_0 = \int_0^t i_g(k,t-s)_0 v(0,s)ds
\]  

(4.55a)

Similar expression can be written for the case when a voltage source is applied at terminal \( z = L \) while terminal \( z = 0 \) is short-circuited.

\[
i(k,t)_L = \int_0^t i_g(k,t-s)_L v(L,s)ds
\]  

(4.55b)

Equations (4.55a) and (4.55b) provide the basis to effectively model a lossy transmission line in the time domain. It appears that all one must do is to short circuit the relevant terminals, set the other voltage to unity and compute \( I(0) \) and \( I(L) \) respectively from equation (4.49) at different frequencies. An inverse Fourier Transform is then performed numerically for \( I(0) \) and \( I(L) \) to obtain the impulse response in time domain. The lossy transmission line is a linear system, the general current - voltage relationship can be written as:

\[
i(k,t) = \int_0^t \left( i_g(k,t-s)_L v(L,s) + i_g(k,t-s)_0 v(0,s) \right)ds = \sum_{T} \int_0^t i_g(k,t-s)_0 v(T,s)ds
\]  

(4.56)

where \( T = 0 \) and \( L \). Equation (4.56) can be generalized to multiconductor system as will be seen in next chapter. It must be mentioned that for the frequency domain counterpart of \( i(t) \) or \( v(t) \) to exist, the function must be time limited e.g. \( i(t) \rightarrow 0 \) as \( t \rightarrow \infty \), otherwise the continuous Fourier transform would have problem converging. Moreover numerical Fourier Transform and inverse Fourier Transform imply the time and frequency domain signals are discretized and analysis is done at discrete points. Such discretization will introduce inherent periodical repetition in the waveform and requires the signal to be both time and
frequency limited. This would naturally imply that the any multiple reflection in the transmission line must dies off after a few transit time. A short circuited or opened end transmission line would have high reflection coefficient and this will render the settling time of the signal very long (since losses in transmission line is very small), resulting in high bandwidth requirement in the frequency domain. A method proposed by Djordjevic et al 1987 is to artificially insert at both ends of the transmission line terminals a linear frequency independent resistance so as to significantly reduce reflections from both ends. This will enable the transient response of the transmission line to die off faster. The value of the resistance is taken as the lossless intrinsic impedance \( R_c \) of the transmission line concerned. This new lossy transmission line model augmented by artificial resistance is then analyzed by the method shown above. To restore the original characteristics, series resistance of value \(-R_c\) will be added to the augmented network. Consider Figure 4.5 below, let the voltage between the two series resistance be the virtual voltage \( v_v(k,t) \).

![Figure 4.5 - Augmented lossy transmission line.](image)

Replacing the integration with summation in equation (4.56):

\[
i(k,t) \equiv \sum_{p=0}^{q} i_g(k,q-p) v_v(0,p) \Delta t + \sum_{p=0}^{q} i_g(k,q-p) v_v(L,p) \Delta t \quad (4.57)
\]

\( k = 0 \) or \( L \). Noting that \( i_g(0,0) = i_g(L,0) = 0 \) because of the time delay in transmission line, equation (4.57) is reordered as below:

\[
i(k = 0, t = q \Delta t) \equiv i_v(k = 0,0) v_v(0,q) \Delta t + \sum_{p=0}^{q-1} i_g(k = 0,q-p) v_v(k = 0,p) \Delta t + \sum_{p=0}^{q-1} i_g(k = 0,q-p) v_v(k = L,p) \Delta t
\]

\( (4.58a) \)
\[ i(k = L, t = q \Delta t) \equiv i_k^i(k = L, 0), v_x(k = L, q) \Delta t + \sum_{p=0}^{q-1} i_k^i(k = L, q - p) v_x(k = 0, p) \Delta t \]

\[ + \sum_{p=0}^{q-1} i_k^i(k = L, q - p) v_x(k = L, p) \Delta t \]  

(4.58b)

It is seen that the current terms depend on the instantaneous voltage at both \( z = 0 \) and \( z = L \) terminals and the voltages in the past. The terms in summation comprise responses due to voltages in the past, i.e. second and third terms of equations (4.56a) and (4.56b) can be grouped together. Equations (4.58a) and (4.58b) can be written in shorter form as:

\[ i(0, t = q \Delta t) = G_{0L} v_x(0, q \Delta t) + i(0, (q - 1) \Delta t) \]  

(4.59a)

\[ i(L, t = q \Delta t) = G_{L0} v_x(L, q \Delta t) + i(0, (q - 1) \Delta t) \]  

(4.59b)

Note that since the transmission line is symmetrical,

\[ G_{L0} = i_x(0, 0) \Delta t = G_{0L} = i_x(L, 0) \Delta t \]  

(4.59c)

Inverting equation (4.59a):

\[ v_x(0, q \Delta t) = G_{0L}^{-1} i(0, q \Delta t) - G_{0L}^{-1} i(0, (q - 1) \Delta t) \]  

(4.60)

and the real voltage at the transmission line port \( z = 0 \) is given as:

\[ v(0, q \Delta t) = v_x(0, q \Delta t) + R_c i(0, q \Delta t) \]

\[ = \left[ R_c + G_{0L}^{-1} \right] i(0, q \Delta t) - G_{0L}^{-1} i(0, (q - 1) \Delta t) \]  

(4.61a)

Similar operation on equation (4.56b) would yield:

\[ v(L, q \Delta t) = v_x(L, q \Delta t) + R_c i(L, q \Delta t) \]

\[ = \left[ R_c + G_{L0}^{-1} \right] i(L, q \Delta t) - G_{L0}^{-1} i(L, (q - 1) \Delta t) \]  

(4.61b)

Therefore the steps in creating a time domain model for a lossy transmission line consist of finding the \( R, L, C, G \) parameters of the line as a function of frequency. The augmented model as shown in Figure 4.5 is formed and the impulse current responses at the two virtual terminals are determined. Then calculate \( G_{L0} \) and finally obtain a complete time domain description of the system as in equations (4.61a) and (4.61b). The term \( R_c + G_{L0}^{-1} \) can be considered as the dynamic input resistance of the transmission line. A model representing the expressions in equation (4.61) is shown in Figure 4.6. This model can be incorporated into SPICE simulation routine.
Figure 4.6 - Time domain model of transmission line.

Care must taken when using the model in Figure 4.6 as the voltage source $G_{0L}^{-1}i(k,(q-1)\Delta t)$ is dependent on time. Most commercial SPICE circuit simulator has provision for performing simulation for single lossy transmission line. For example in PSPICE version 6.1 a single lossless and lossy transmission model using the provisions in Section 4.5 is supported.

4.6 Simulation Example

This example uses a co-axial transmission line of length 1.0 meter and the followings sizes and parameters.

Dielectric material of co-axial cable: Teflon
Conductivity of co-axial cable conductor = 5.0x10^7 S
inner diameter of co-axial cable = 1.00mm
outer diameter of co-axial cable = 3.48mm
length L = 1.00m
$Z_g = 30\Omega$  $Z_L = 70\Omega$

Figure 4.7 - Model of the simulation example.
Both frequency and time domain simulations are shown. The time domain simulation is performed by first determining the impulse response for v0(t) and vL(t) of the system. A numerical convolution is then carried out. The forcing function is a square pulse with amplitude of 1.0 Volts and duration of 12.0ns. In this example the signal is assumed to be periodic with a period of 8 times the transit time to across the transmission line. Multiple reflection is assumed to die off after 8 transitions along the line. The forcing functions Vs(t) and δ(t) are periodic function with T = 8Td. Consequently Fourier series representation is used. Values for R,L,C,G for the transmission line are determine from equations (4.21a) to (4.21d). For wide bandwidth signal Fourier series representation is preferred to Fourier transform due to accuracy of performing numerical integration. In the case of Fourier transform a large number of discrete frequency points would have to be used to achieve acceptable accuracy, whereas up to 50 harmonics in the Fourier series will yield accurate results. The transfer functions for voltage Hv(z,ω) and current Hi(z,ω) along the axis of the transmission line are determined using equation (4.49) as:

\[ Hv(z,\omega) = \frac{Zo(\omega)}{Zg(\omega) + Zo(\omega)} \cdot \frac{\exp(-\gamma(\omega)L)}{1 - \Gamma_s(\omega) \Gamma_L(\omega)} \cdot \left( e^{\gamma(\omega)z} + \Gamma(\omega) e^{-\gamma(\omega)z} \right) \]  

(4.62a)

\[ Hi(z,\omega) = \frac{1}{Zg(\omega) + Zo(\omega)} \cdot \frac{\exp(-\gamma(\omega)L)}{1 - \Gamma_s(\omega) \Gamma_L(\omega)} \cdot \left( e^{\gamma(\omega)z} + \Gamma(\omega) e^{-\gamma(\omega)z} \right) \]  

(4.62b)

where \( \Gamma_s(\omega) = \frac{Z_L - Zo}{Z_L + Zo} \) and \( \Gamma_L(\omega) = \frac{Z_L - Zo}{Z_L + Zo} \) and Zo(ω) = \( \sqrt{\frac{R+j\omega L}{V+j\omega C}} \).  

(4.62c)

### 4.6.1 Performing Numerical Convolution

Assuming the impulse response to be repetitive with period T, the approximate expressions for time domain voltage impulse response and time domain convoluton are given respectively by:

\[ hv(z,t) = \sum_{k=-N}^{N} \frac{1}{T} Hv(z,\frac{2\pi k}{T}) e^{-\frac{2\pi k t}{T}} \]  

(4.63a)
\[
v(z,t) = \int_{0}^{t} h(v(z,t - \tau))v_{\text{source}}(\tau)d\tau, \quad 0 < t < T
\]
\[
\cong \sum_{i=0}^{m} h(v((m-i)\Delta t))v_{\text{source}}(i\Delta t)\Delta t, \quad 0 < m < n - 1
\]
\[
\Delta t = \frac{T}{n}
\]
In equation (4.63a) N is the highest harmonics while the rectangular rule is used to approximate the integration operator in equation (4.63b), n is the number of partition along the time axis. Figure 4.8 shows the impulse responses for the voltage at source and load ends of the transmission line. \(v_0(t)\) and \(v_L(t)\) correspond to source and load ends voltages respectively due to a unit impulse excitation. It is obvious that the impulse responses contain delta function.

\[\text{Load end}\]
\[\text{Source end}\]

**Figure 4.8** - Time domain impulse response for \(v_0(t)\) and \(v_L(t)\).

Since the impulse response for voltages contain delta function, great care must be taken to ensure acceptable error. The integer N and the number of partition n must be related by :
\[
n = 2N + 1 \cong 2N
\]
The relation (4.64) is derived as follows. Consider an impulse response as depicted in Figure 4.9. The time domain convolution of this function with an excitation of \(e(t) = 1\) is just a delayed unit step function.
Figure 4.9 - Impulse response and convoluted response.

The delayed unit step function can be written as:

\[ U(t - t_o) = \int_0^T h(\tau) e(t-\tau) d\tau = \int_0^T h(\tau) d\tau \]  

(4.65)

Using rectangular rule:

\[ U(m\Delta t) \equiv \sum_{i=0}^{\infty} h(i\Delta t) \Delta t \]  

(4.66a)

\[ h(t) \equiv \begin{cases} \frac{1}{T} \sum_{k=-N}^{N} e^{\frac{2\pi ik(t-t_0)}{T}}, & \text{for } 0 < t < T \\ 0, & \text{otherwise} \end{cases} \]  

(4.66b)

\[ \Delta t = \frac{T}{n}, \ t_o = d\Delta t \]  

(4.66c)

Since the contribution to convolution mostly originates from region near the impulse:

\[ U(m\Delta t) \equiv \begin{cases} h(d\Delta t) \Delta t, & m \geq d \\ 0, & m < d \end{cases} \]  

(4.67a)

and

\[ h(d\Delta t) \Delta t = h(t_o) \Delta t = \frac{1}{n} \sum_{k=-N}^{N} \left( \frac{2N+1}{n} \right) \]  

(4.67b)

Equation (4.67b) will only result in a step of unity if \( n = 2N + 1 \), otherwise there will be an error in the level. Thus in general it is observed that to achieve
maximum degree of accuracy with rectangular numerical integration for time
domain convolution, the time axis must be sufficiently partitioned to \( n \) intervals.
Furthermore \( n \) must be related to the highest harmonics \( N \) by equation (4.64). The
results in Figure 4.10 and Figure 4.11 illustrates the idea. The time axis is
partitioned into \( n = 100 \) segments for sufficient accuracy. Figure 4.10 shows the
comparison of time domain result using inverse Fourier series and convolution
analysis for \( v_0(t) \) when \( N = 50 \). Figure 4.11 shows the comparison of time domain
result using frequency domain analysis and convolution analysis when \( N = 30 \) and
\( N = 70 \) respectively. Note the error in d.c. level for \( v_0(t) \) for results obtained using
convolution analysis in Figure 4.11. Insufficient and excess \( N \) as compared to
\( n/2=50 \) reduce and increase the d.c. level incorrectly.

**Figure 4.10** - When \( N = 50 \) and \( n = 100 \). \( v_{10}(t) \) = frequency domain result, \( v_0(t) \) =
time domain convolution analysis results.
Figure 4.11 - When \( n = 100 \). \( v_{10}(t) \) = frequency domain result, \( v_0(t) \) = time domain convolution analysis result (\( N = 70 \)), \( v_1(t) \) = time domain convolution analysis result (\( N = 30 \)).

4.6.2 Comparison of Results

Finally Figure 4.12A and Figure 4.12B compares the results obtained using Fourier series analysis and time domain convolution of impulse response for the system shown in Figure 4.7. Similar excitation \( V_{source}(t) \) is used for both Fourier Transform and Time Domain Convolution approaches. As can be seen both results agrees well if the requirement of equation (4.64) is complied and sufficient partitioning of the time axis is carried out.

Figure 4.12A - Result using Fourier series analysis \( v_{10} = v_0(t) \) and \( v_{1L} = v_L(t) \). \( V_{source} = V_s(t) \).
Figure 4.12B - Result using time domain convolution analysis $v_0 = v_0(t)$ and $v_L = v_L(t)$. $V_{source} = V_s(t)$. 