CHAPTER 3 Derivation of Equivalent Circuit Model for PCB Trace Discontinuities

3.1 Interconnection Discontinuity

In a Printed Circuit Board environment, the common encountered interconnection discontinuities are shown below in Figure 3.1:

![Figure 3.1 - Common discontinuity structures in PCB assembly system.](image)

Discontinuities in interconnection are usually the result of change in the transmission geometry to accommodate component layout on the printed circuit board. These types of discontinuities are usually called passive discontinuities. Introduction of passive discontinuities will distort the uniform electromagnetic field present in the infinite length transmission line. This distortion of field in the vicinity of the discontinuity can be viewed as superposition of induced higher order modes to match the boundary conditions of the discontinuity. Most of these higher order modes are evanescent or non-propagating and they attenuate rapidly from the discontinuity. In practice at a sufficient distance of a few operating wavelengths from the discontinuity, the field of these higher order modes will virtually vanish, leaving the original dominant mode field unperturbed. Thus these higher order fields are usually known as local fields. The local fields are usually reactive since loss due to dielectric is negligible and the discontinuity is normally made of excellent conducting material. Consider the transmission line
bend shown in Figure 3.2. Assuming the reference plane AA’ and BB’ are taken sufficiently far away. The region between AA’ and BB’ can be represented by a two ports network as shown in Figure 3.2:

![Diagram of transmission line bend and equivalent two port network.](image)

**Figure 3.2** - Transmission line bend and equivalent two port network.

Equivalent circuit can be derived to approximate the characteristics of the two port network as closely as possible. Different approaches can be used to model the passive discontinuity, notably analytical method, measurement based method and field solution method using electromagnetic simulation software employing Finite Element Methods or Methods of Moment. The two port networks for discontinuity usually consists of both lumped and distributed circuit elements. In this chapter the writer attempts to compare and contrast the various approaches available in the open literature. An experimental PCB with trace discontinuities such as bend, vias and ground plane gap is constructed. TDR measurements are made on these discontinuities and comparison between simulated waveforms of equivalent circuit models and measurements are provided with the aim of validating the corresponding models.

### 3.2 Analytical Based Method
Analytical method is usually applied when only first order approximation is required (zeroth order approximation considers the discontinuity to be non-existent). The discontinuity is usually approximated as discrete elements such as resistor, inductor and capacitor. An example is a pad across a microstrip line as in Figure 3.3. The pad will be considered as a shunt capacitance to ground where

\[ C_{pad} = \frac{\varepsilon_r \varepsilon_0 A}{d}, \quad A = \text{area of pad} \quad (3.1) \]

\[ L = 0.129 h \left[ \ln \left( \frac{d}{h} \right) + 1 \right] \quad (3.2) \]

where \( h \) is the effective height of the via and \( d \) is the diameter of the via.
3.3 Field Solution Method

A more accurate model using electromagnetic field solution software and measurement takes into account the excess charge and excess magnetic flux at the vicinity of the discontinuity. Referring to Figure 3.3 again, at region far away from reference plane AA’ and BB’, the electric charge distribution and magnetic flux linkage between trace and ground plane will approach the configuration of an infinite transmission line. Between AA’ and BB’, in addition to the infinite transmission line charge and flux distribution, there are charge and flux distributions due to higher order modes. These charge and flux will complement the infinite transmission line distribution and are known as excess charge and excess flux. It is noted that the excess distribution will increase or decrease the original distribution depending upon the geometry of the interconnection discontinuity. Associated with the excess charge, a lumped equivalent capacitance can be assigned at the discontinuity (Benedek and Silvester, 1972 and 1973). Note that the value of this capacitance can be positive or negative depending on the polarity of the excess charge. Similarly a lumped equivalent inductor is assigned for excess flux at the discontinuity (Gopinath and Thomson, 1975). The excess charge and flux are determined from electric and magnetic field between the reference planes AA’ and BB’. Solution for the electric and magnetic fields are computed using numerical methods such as Finite Element Method or Method of Moments. Transmission line discontinuity is usually modeled as a hybrid of

Figure 3.4 - Via discontinuity and first order equivalent circuit approximation.
transmission lines and lumped RLCG discrete elements. Since this type of discontinuity is usually symmetrical (both sides of the discontinuity are transmission lines) and virtually lossless, a $\pi$ or T second order equivalent circuit can be constructed as shown in Figure 3.5, (Collin 1992).

![Figure 3.5](image-url) - Second order equivalent circuit for discontinuity.

### 3.3.1 Outline for Deriving Approximate Equivalent Model

An outline for determining the equivalent values of L and C is given following the procedures proposed by Benedek et al 1972 and 1973, Thomson and Gopinath 1975. Assuming the discontinuity to be lossless with perfect conductor, Figure 3.6 shows a representation of the model:

![Figure 3.6](image-url) - Representation of transmission line discontinuity.

The component values can be obtained through a de-embedding process outlined below:

- First solves the entire three dimensional static $\mathbf{H}_{3D}$ and $\mathbf{E}_{3D}$ fields in the entire problem region.
• Consider the two ends as uniform transmission line, solve the two dimensional static $H_{2D}$ and $E_{2D}$ fields.

• Let $L_1 = \text{per unit length inductance of transmission line } T_1$, $C_1 = \text{per unit length capacitance of transmission } T_1$. Similarly $L_2$ and $C_2$ are the per unit length parameters for $T_2$. Determine these parameters from $H_{2D}$, $E_{2D}$ and equations (2.13), (2.14).

• The total electric field energy stored within the problem region $V$ corresponds to combination of $C_1, C_2$ and $C_{\text{dis}}$, the discontinuity capacitance due to excess charge.

$$C_{\text{total}} = \iiint E_{\text{total}} \cdot D_{\text{total}} dx dy dz = C_1 l_1 + C_2 l_2 + C_{\text{dis}} \quad (3.3)$$

• The total magnetic field energy stored within the problem region $V$ corresponds to $L_1, L_2$ and $L_{\text{dis}}$, the discontinuity inductance due to excess flux.

$$L_{\text{total}} = \iiint H_{\text{total}} \cdot B_{\text{total}} dx dy dz = L_1 l_1 + L_2 l_2 + L_{\text{dis}} \quad (3.4)$$

• From (3.3) and (3.4), $L_{\text{dis}}$ and $C_{\text{dis}}$ for the discontinuity can be determined.

Note that in most instances this direct method of deriving second order approximation for the equivalent circuit of discontinuity involves subtraction of two values, which are nearly equal, resulting in substantial error. An indirect method for computing $L$ has been given by Thomson and Gopinath 1975, using magnetic vector potential. The procedure as outlined above has been incorporated in commercial electromagnetic field simulator software (Ansoft 1993).

### 3.4 Time Domain Measurement Method

Time Domain Reflectometry (TDR) as discussed in Chapter Two is another effective way to derive the approximate lossless model for transmission line discontinuities. Here the method of Section 2.3 by performing integration (zeroth moment) on the reflected voltage response of the discontinuity cannot be applied as the other end of the discontinuity is not opened or shorted. Another approach
of TDR modeling based on analyzing the impedance profile of the transmission line is presented here (Hayden et al 1990 and Jong et al 1993). A transmission line interconnection system can be considered as a one dimensional system with the local impedance being a function of distance along the length of interconnection. Here for simplicity of analysis the interconnection is assumed to be lossless. Very recently some researchers have reported algorithms which are able to determine impedance profile of lossy transmission line system using segmented transmission lines and lumped resistors, (Jong et al 1994). Derivation of the lossless impedance function or impedance profile through TDR measurement is described in Jong et al 1993 and Tektronix Application Note 1993.

The idea is to consider an interconnection such as a transmission line system, including the discontinuities as consisting of many small transmission lines segments, each with characteristic impedance $Z_c$. This concept is shown in Figure 3.7. The time $\Delta t$ required for an electromagnetic wave to transverse each segment is the same. Therefore it is evident the physical lengths $l_i$ of the segments are not equal in general. Assuming the segments to be lossless, a real impedance $R_i$ can be assigned to each segments with:

$$R_i = \frac{L_i}{\sqrt{C_i}}$$

(3.5)

where $L_i$ and $C_i$ are the local per unit length inductance and capacitance of segment $i$ respectively. Corresponding to each intersection between segments $i$ and $i+1$, a reflection coefficient $\rho$ and transmission coefficient $\tau$ can be defined.
Figure 3.7 - Assigning an interconnection system to a series connection of transmission line segments.

The reflection and transmission coefficients can be defined for the forward and backward incident waves. Using “-” to denote forward direction and “+” to denote backward direction, let the reflection coefficient be $\rho^\pm$ and transmission coefficient $\tau^\pm$ in time domain. Figure 3.8 illustrates the convention adopted. Furthermore:

$$\rho_i^- (t) = \frac{V_i^-_{\text{ref}} (t)}{V_i^-_{\text{inc}} (t)} \quad (3.6a)$$

$$\tau_i^- (t) = \frac{V_i^-_{\text{tran}} (t)}{V_i^-_{\text{inc}} (t)} = 1 + \rho_i^- (t) \quad (3.6b)$$

$V_i^-_{\text{ref}} (t)$ is the reflected voltage, $V_i^-_{\text{tran}} (t)$ is the transmitted voltage and $V_i^-_{\text{inc}} (t)$ is the incidence voltage. From equation (2.24c) when $Z_L$ is purely resistive:

$$\rho_i^- (t) = \frac{R_{i+1} - R_i}{R_i} \quad (3.7a)$$

$$\tau_i^- (t) = \frac{2R_{i+1}}{R_{i+1} + R_i} \quad (3.7b)$$

For a backward traveling wave, $\rho_i^+$ and $\tau_i^+$ are related to equations (3.7a) and (3.7b) as:

$$\rho_i^+ (t) = -\rho_i^- (t) \quad (3.8a)$$

$$\tau_i^+ (t) = 1 - \rho_i^- (t) \quad (3.8b)$$

Figure 3.8 - Convention for reflection and transmission coefficients
Assuming $R_i$ is known, $R_{i+1}$ can be determined from (3.7a) as:

$$R_{i+1} = \frac{1 + \rho \cdot i \cdot (t)}{1 - \rho \cdot i \cdot (t)}$$  \hfill (3.9)

Consider a forward directed step pulse $U(t)$ incident on a system similar to Figure 3.7. The step pulse can be approximated as a superposition of overlapping narrow rectangular pulses of width $2\Delta t$ (Figure 3.9A). Snapshots of the reflected and transmitted waves at three instances at $2\Delta t$, $4\Delta t$ and $6\Delta t$ are shown in Figure 3.9B.

**Figure 3.9A** - Approximation of step source with discrete rectangular pulses

**Figure 3.9B** - Snapshots of the incident and reflected waveforms at three different instances.

From Figure 3.9B, the reflected voltage wave due to the overlapping rectangular pulses will be of similar form, that is comprising of overlapping narrow rectangular pulses of width $2\Delta t$. Examining the three snapshots will enable us to write the following equations:

$$\text{Ref}[1] = \rho \cdot \text{Inc}[1]$$  \hfill (3.10a)
\[
\text{Ref}[2] = \tau_i \tau^*_i \rho^{-2} \text{Inc}[1] + \rho^i \text{Inc}[2]
\]

\[
= \left(1 - \left(\rho^i\right)^2\right) \rho^{-2} \text{Inc}[1] + \rho^i \text{Inc}[2]
\]  \hspace{1cm} (3.10b)

\[
\text{Ref}[3] = \left(\tau_i \tau^*_i \tau^*_j \rho^{-3} + \tau_i \tau^*_i \rho^{-2} \rho^j\right) \text{Inc}[1] + \tau_i \tau^*_i \rho^{-2} \text{Inc}[2] + \rho^j \text{Inc}[3]
\]

\[
= \left(\left(1 - \rho^i\right)^2 \left(1 - \rho^j\right)^2\right) \rho^{-3} \text{Inc}[1] + \left(1 - \left(\rho^i\right)^2\right) \rho^{-2} \text{Inc}[2]
\]

\[+ \rho^j \text{Inc}[3]\]

(3.10c)

The reflected pulses for Ref[4], Ref[5] ... etc. can be written out in a similar manner. A matrix equation can be formed:

\[
\begin{bmatrix}
\text{Ref}[1] \\
\text{Ref}[2] \\
\text{Ref}[3] \\
\vdots \\
\text{Ref}[n]
\end{bmatrix}
= 
\begin{bmatrix}
c_1 & 0 & 0 & \cdots & 0 \\
c_2 & c_1 & 0 & \cdots & 0 \\
c_3 & c_2 & c_1 & \cdots & \text{Inc}[3] \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_n & c_{n-1} & \cdots & c_2 & c_1 & \text{Inc}[n]
\end{bmatrix}
\]

(3.11)

Coefficients \(c_i\) in the matrix of equation (3.11) can be determined using the reflection diagram of Figure 3.9B. For instance \(c_4\) is given by:

\[
c_4 = \left(\rho^{-2}_2\right)^3 \left(\rho^{-2}_1\right)^2 \left(1 - \left(\rho^{-2}_1\right)^2\right) - \left(\rho^{-2}_3\right)^2 \rho^{-2}_2 \left(1 - \left(\rho^{-2}_2\right)^2\right) \left(1 - \left(\rho^{-2}_1\right)^2\right) +
\]

\[
\left(1 - \left(\rho^{-2}_1\right)^2\right) \left(1 - \left(\rho^{-2}_2\right)^2\right) \left(1 - \left(\rho^{-2}_3\right)^2\right) \rho^{-2}_4
\]

(3.12)

Both the values of Ref[i] and Inc[i] can be obtained by measurement, for instance using digital sampling oscilloscope, and values of \(c_i\) can be solved. This essentially means that the reflected voltage waveform is also discretized into overlapping rectangular pulses. Coefficients in the square matrix of equation (3.11) are then solved:

\[
c_1 = \frac{\text{Ref}[1]}{\text{Inc}[1]} \quad c_2 = -\frac{1}{\text{Inc}[1]} \left(\text{Ref}[2] - \text{Inc}[2] c_1\right) \text{ etc.}
\]

(3.13)

Once all the coefficients \(c_i\) are determined, value of \(\rho^i\) for each segment \(i\) can be calculated from equation (3.12) and lossless characteristic impedance of the segments \(R_0\), \(R_1\) ... \(R_n\) can be determined from equation (3.9). A plot of the
impedance value as a function of time or distance along the interconnection is called the *impedance profile* of the system. Resolution of the sampler will determine the values of n and Δt. Larger n and smaller Δt will result in more segmentation of a transmission line system, hence a more accurate model for the transmission line system. In this respect larger n will also result in more complex model, thus accuracy is traded with complexity. However it must be pointed out that the ultimate resolution is dictated by the transition rate of the incident step, τr. Any structure smaller than τr/2 will not be resolved from the reflection. Due to the cumulative manner of solving (3.13) in which higher order coefficient c_i is dependent on lower order coefficients c_1, c_2, ..., c_{i-1}, any error in the earlier coefficients for instance c_1 will ‘snowball’ into large error for higher coefficients. Earlier coefficients are vulnerable to noise since the amplitude of the rectangular pulses for Inc[1], Inc[2] ..., Ref[1], Ref[2] ... are small. One way to overcome this is to limit the smallest value of Inc[j] by applying signal preprocessing to artificially modify the incidence voltage signal (Tektronix Application Note 1993).

### 3.4.2 Deriving Approximate Model for Discontinuity

The method described above is only valid when the transmission line system is not coupled to any other object. If substantial coupling does occur then some of the incident energy would be loss through coupling and the impedance profile obtained using reflected voltage will suffer from excessive error. By comparing the impedance profile of a controlled transmission line system and an impedance profile of transmission line with discontinuity, variation due to discontinuity can be pinpointed. The discontinuity is then modeled as a sequence of short transmission lines by partitioning the location where variation in impedance occurs (see Section 3.5). Alternatively we could choose to use symmetry π or T model of Figure 3.5. The equivalent L and C values can be derived from the distributed inductance and capacitance of the impedance profile. Consider a
segment \( j \) of transmission line with length \( \Delta l_j \) and characteristic impedance \( R_j \). If \( \Delta t_j \) is the time delay for the transmission line section then:

\[
\Delta t_j = \frac{\Delta l_j}{v_j} = \Delta l_j \sqrt{L_j C_j}
\]

(3.14)

where \( L_j \) and \( C_j \) are the inductance and capacitance per unit length of segment \( j \). Observe that the product

\[
Z_j \Delta t_j = \sqrt{L_j C_j} \Delta l_j = L_j \Delta l_j
\]

(3.15a)

gives the inductance of segment \( j \). Hence if given a length of transmission line \( l \), knowing that the time corresponding to two ends of the line are \( t_1 \) and \( t_2 \), total inductance of the line can be expressed as:

\[
L = \int_{t_1}^{t_2} Z(t) dt
\]

(3.15b)

Using similar argument the total capacitance between two ends of the line can be expressed as:

\[
C = \int_{t_1}^{t_2} \frac{1}{Z(t)} dt
\]

(3.16)

The introduction of a discontinuity alters the effective impedance along a transmission line. By restricting the time limits to starting and ending of the discontinuity in an impedance profile, the effective inductance and capacitance of the discontinuity can be determined. This in turn enables the construction of symmetry equivalent model as in Figure 3.5. However the symmetry \( \pi \) or \( T \) model derived in this manner is usually less accurate than using a sequence of transmission lines. Comparison between segmented transmission line model and symmetry \( T \) model is performed in Section 3.5.2.

### 3.5 Frequency Domain Measurement Method

The most accurate method of deriving equivalent circuit model for interconnection discontinuity is based on measurement of d.c. and S-parameters values \( S_{11} \) and \( S_{12} \) (Sadhir et al 1994). The equivalent circuit model parameters are extracted by
computer optimization which proposes an equivalent circuit using RLCG circuit elements and an attempt to correlate the simulated d.c. and S-parameters with measured data by tuning the values of the RLCG elements. Two approaches are suggested by the writer in this section based on modification of Bandler et al 1986 and 1987 optimization procedures for analog circuit design. Consider a two port network connected at both ends by transmission lines of impedance $Z_{o1}$ and $Z_{o2}$ respectively. The S-parameter representation for the two ports is given by:

\[
b_i = S_{i1}a_1 + S_{i2}a_2 \tag{3.17a}
\]

\[
b_2 = S_{21}a_1 + S_{22}a_2 \tag{3.17b}
\]

\[
b_i = \text{reflected wave at port } i = \frac{V_i + Z_{o1}I_i}{2\sqrt{Re(Z_{o1})}} \tag{3.18a}
\]

\[
a_i = \text{incident wave at port } i = \frac{V_i - Z_{o1}^*I_i}{2\sqrt{Re(Z_{o1})}} \tag{3.18b}
\]

where $i = 1$ or 2. Now consider the impedance representation for the two ports:

\[
V_1 = Z_{i1}I_1 + Z_{i2}I_2 \tag{3.19a}
\]

\[
V_2 = Z_{21}I_1 + Z_{22}I_2 \tag{3.19b}
\]

The linear nature of equations (3.17a) and (3.17b) is attributed to Maxwell’s equations which are linear and satisfy reciprocity relation (Ramo et al 1965). If $Z_{o1}=Z_{o2}=Z_o$, the S-parameters and impedance parameters are related by (Kuo 1966, Vendelin et al 1990):

\[
\overline{S} = \left( \overline{Z} - Z_o \overline{U} \right) \left( \overline{Z} + Z_o \overline{U} \right)^{-1} \tag{3.20}
\]

where $\overline{U}$ is the identity matrix. After solving for $\overline{Z}$:

\[
Z_{i1} = \frac{Z_o (1 + S_{11})(1 - S_{22}) + S_{12}S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{3.21a}
\]

\[
Z_{i2} = \frac{2Z_o S_{12}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{3.21b}
\]

\[
Z_{21} = \frac{2Z_o S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{3.21c}
\]
\[ Z_{22} = Z_o \frac{(1 - S_{11})(1 + S_{22}) + S_{12}S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \]  

(3.21d)

Assuming the material making up the discontinuity are linear, and since Maxwell’s equations satisfy reciprocity theorem, we have:

\[ Z_{12} = Z_{21} \]  

(3.22)

The impedance \(Z_{11}\) and \(Z_{22}\) are equal only when the discontinuity concerned is symmetrical, for example a transmission line bend. To derive an approximate equivalent circuit model for the discontinuity, S-parameters of the discontinuity is first measured up to microwave region. The S-parameters are then converted to impedance parameters for optimization. It is easier to work with impedance parameters as they can be related directly to lumped circuit elements.

The first approach to determine a lumped circuit model which would provide response close to the measured impedance parameters is to use a rational polynomial to approximate the impedance \(Z_{11}, Z_{22}\) and \(Z_{12}\). A network model for the two ports network is shown below Figure 3.10A.

![Figure 3.10 - Symmetry Z model.](image)

By studying the amplitude and phase characteristics of \(Z_{11}\) and \(Z_{12}\), optimized rational polynomials in the form of:

\[ z_{ij}(\omega) = \sum_{k=0}^{K_1} a_{k_1} (j\omega)^{k_1} \sum_{k=0}^{K_2} b_{k_2} (j\omega)^{k_2} \]  

(3.23)

can be assigned to approximate \(Z_{11}\) and \(Z_{12}\). Since most discontinuities are virtually lossless, the impedance functions will only have imaginary components.
Hence $z_{ij}$ in equation (3.23) must be the ratio of even function to odd function or vice versa (Kuo 1965). The optimization problem can be formulated as follows:

Minimize $F(a_1, a_2, \ldots, a_{k_1}, b_{k_2}, b_{k_2}, \ldots, b_{k_2}) = \left[ \sum_r w_r \left[ z_{ij}(\omega_r) - Z_{ijr} \right]^p \right]^{1/p}$ \hspace{1cm} (3.24a)

with the constraints:

$$a_{k_1} \geq 0 \text{ and } b_{k_2} \geq 0$$ \hspace{1cm} (3.24b)

where $r = 0, 1, 2, 3, \ldots, n$ are the sampling points, $Z_{ijr}$ is the measured value at frequency $\omega_r$ and $w_r$ is the weighting constant. The expression in equation (3.24a) is called the Norm $l_p$, and $p$ is an integer. When $p = 2$, this becomes the well known least square minimization problem, and when $p$ approaches infinity, the problem reduces to the min-max optimization problem (Scheid 1990).

Optimization of equation (3.24a) can be done using non-linear programming (Wismer and Chattergy 1978, Russel 1970). Nevertheless it is noted that for $p$ larger than 2 the results of optimization for variables $a_{k_1}$ and $b_{k_2}$ are sensitive to any isolated large deviation of the measurement due to random error. A more efficient method would be to use the $l_1$ ($p = 1$) norm which is insensitive to large isolated error (Bandler et al 1986 and 1987). Equation (3.24a) under $l_1$ norm would be written as:

Minimize $F(a_1, a_2, \ldots, a_{k_1}, b_{k_2}, b_{k_2}, \ldots, b_{k_2}) = \sum_r \left| z_{ij}(j\omega_r) - Z_{ijr} \right|$ \hspace{1cm} (3.25)

However the $l_1$ norm is not easy to minimize since the derivative of $|a_k|$ relative to $a_k$ is either 1 or -1. Specialized optimization method has to be applied, and this is shown in detail in Bandler et al 1987. After the approximate rational polynomials are obtained for $z_{11}(s)$, $z_{22}(s)$ and $z_{12}(s)$, they must be checked to ensure realizability and causality, e.g. the rational polynomials must be positive real functions. Equivalent RLCG networks can then be derived for the model proposed in Figure 3.10 using continuous division synthesis method (Kuo 1966). It is evident that certain discontinuity, such as that encompasses long transmission line sections will lead to non-realizable network.
The second approach is more direct and easier to implement although less accurate. An example of this approach will be shown at Section 3.5.4. Considering only localized and lossless interconnect discontinuity, the equivalent circuit shown in Figure 3.11 is assumed to approximate a non-symmetrical discontinuity under quasi-TEM condition. The accuracy of this model is dependent on the dimension of the discontinuity. For PCB trace bend, it is good until 10GHz while for large discontinuity such as connector to trace interface, upper valid frequency is only 5GHz. As the operating frequency increases, shortest wavelength gradually becomes comparable to the discontinuity, and the addition of distributed components into the equivalent circuit of Figure 3.11 is necessary.

![Figure 3.11 - Assumed discontinuity equivalent circuit.](image)

The impedance parameters for the equivalent circuit in Figure 3.11 are:

\[
\begin{align*}
Z_{11}(s) &= sL_1 + \frac{1}{sC} \\
Z_{22}(s) &= sL_2 + \frac{1}{sC} \\
Z_{12}(s) &= Z_{21}(s) = \frac{1}{sC}
\end{align*}
\]

(3.26a)

(3.26b)

(3.26c)

Using equation (3.20), the theoretical S-parameters can be determined from equation (3.26a) to (3.26c) through:

\[
\begin{align*}
S_{11} &= \frac{(Z_{11} - Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}} \\
S_{22} &= \frac{(Z_{11} + Z_o)(Z_{22} - Z_o) - Z_{12}Z_{21}}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}
\end{align*}
\]

(3.27a)

(3.27b)
\[ S_{13} = \frac{2Z_o Z_{12}}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12} Z_{21}} \]  
(3.27c)

where \( Z_o \) is the reference impedance. The optimization problem can be formulated using least square criteria \((l_2)\) by comparing real and imaginary part of \( S_{11}, S_{22} \) and \( S_{12} \) with measured \( S_{11}, S_{22} \) and \( S_{12} \) respectively.

Minimize :

\[
F(L_1, L_2, C) = \sum_r w_1 \left[ \text{Re} \ S_{11} (j \omega_r) - \text{Re} \ S_{11} \right]^2 + \sum_r w_2 \left[ \text{Re} \ S_{22} (j \omega_r) - \text{Re} \ S_{22} \right]^2 \\
+ \sum_r w_3 \left[ \text{Re} \ S_{12} (j \omega_r) - \text{Re} \ S_{12} \right]^2 + \sum_r w_4 \left[ \text{Im} \ S_{11} (j \omega_r) - \text{Im} \ S_{11} \right]^2 \\
+ \sum_r w_5 \left[ \text{Im} \ S_{22} (j \omega_r) - \text{Im} \ S_{22} \right]^2 + \sum_r w_6 \left[ \text{Im} \ S_{12} (j \omega_r) - \text{Im} \ S_{12} \right]^2 
\]

(3.28)

where \( w_k, k = 1,2...6 \) is a suitably chosen weighting constant. The necessary conditions for stationary point of \( F(L_1, L_2, C) \) are :

\[
\frac{\partial F}{\partial L_1} = 0 \quad (3.29a)
\]

\[
\frac{\partial F}{\partial L_2} = 0 \quad (3.29b)
\]

\[
\frac{\partial F}{\partial C} = 0 \quad (3.29c)
\]

Analytical expressions for equations (3.29a) to (3.29c) are extremely complicated and it is often more convenient to obtain a solution for \( L_1, L_2 \) and \( C \) through iteration method such as Method of Steepest Descent. Values for the partial derivatives are computed using perturbation method. To check that solution of equations (3.29a) to (3.29c) do provide a strong minimum point, a sufficient condition is that the Hessian matrix \( \overline{H} \) must be positive definite at the stationary point (Chapter 2, Wismer and Chattergy 1978). There are numerous methods to check for the positive definiteness of Hessian matrix ranging from finding the eigenvalues to using Sylvester Theorem (Ayres 1963, Wismer and Chattergy 1978). Applying Sylvester Theorem :
\[
H = \begin{bmatrix}
\frac{\partial^2 F}{\partial L_1 \partial L_1} & \frac{\partial^2 F}{\partial L_1 \partial C} & \frac{\partial^2 F}{\partial L_1 \partial R} \\
\frac{\partial^2 F}{\partial L_2 \partial L_1} & \frac{\partial^2 F}{\partial L_2 \partial C} & \frac{\partial^2 F}{\partial L_2 \partial R} \\
\frac{\partial^2 F}{\partial L_3 \partial L_1} & \frac{\partial^2 F}{\partial L_3 \partial C} & \frac{\partial^2 F}{\partial L_3 \partial R}
\end{bmatrix}
\]  
\quad \text{(3.30)}

Sylvester Theorem states that for a symmetrical matrix to be positive definite, all its principal minors must be positive. Thus Sylvester Theorem translates into:

\begin{align*}
h_{11} &= \frac{\partial^2 F}{\partial L_1 \partial L_1} > 0 \quad \text{(3.31a)} \\
h_{22} &= \frac{\partial^2 F}{\partial L_2 \partial L_2} > 0 \quad \text{(3.31b)} \\
h_{33} &= \left| H \right| > 0 \quad \text{(3.31c)}
\end{align*}

The equivalent circuit model of Figure 3.11 is suitable for simple discontinuities which include transmission line bend, step, and ground plane gap etc. When the model cannot approximate a discontinuity sufficiently the values of either \(L_1\), \(L_2\) or \(C\) will become negative or no minimum point will be found. More complicated equivalent circuit must be used to adequately model the discontinuity. Having more elements will allow for greater degree of freedom in optimizing the measured and theoretical response. Carefully chosen equivalent circuit model will enable highly accurate representation of the discontinuity in the range of frequency under consideration. The accuracy of measurement-based methods depends upon the accuracy of the measurement, calibration techniques and calibration standards. Conventional OSM (open, short, match) calibration for vector network analyzer is not applicable for small planar structures in PCB. More recent calibration methods such as line-reflect match (LRM) and thru-reflect-line must be employed (Wilton 1991, Sadhir and Bahl 1994). This entails building test structures into the PCB under test and greatly increases the difficulty of frequency domain measurement approach as opposed to using TDR or field solution method.

3.5 Measurement Example and Validation of Models
In this example equivalent circuit models are derived for a number of striplines punctuated with vias, 90 degrees bend and ground plane gap. The equivalent circuit models for discontinuities are frequency dependent, however this is often ignored. This approximate equivalent model is still sufficiently accurate for the purpose of high speed digital design. Some discussion on validating the equivalent models is provided in Section 3.5.1. Section 3.5.2 presents an example of modeling of a via using field solution method. Section 3.5.3 presents examples of modeling a via, ground plane gap in transmission line and transmission line bend using TDR measurement. Finally section 3.5.4 presents a modeling example using S-parameter measurement for SMA to PCB connector.

### 3.6.1 Validation of Models

One way to validate the accuracy of the model is to incorporate the discontinuities in a transmission line type resonator. The resonator is excited through weak coupling by another transmission lines. Resonant frequencies of the resonator are measured and compared with values predicted using models created with section of transmission lines and the approximate discontinuity models (Hoefer et al 1975, Douville et al 1978, Easter 1975 and Slobodnik et al 1994). For instance Slobodnik and Webster 1994 shows that error of less than ±5% between measured and calculated resonance frequency for microstrip resonator having bends can be achieved within 18-60GHz for carefully planned measurement algorithm and robust discontinuity model extraction software using field solution method. However the major drawback in using frequency domain measurement is that precision calibration structures have to be built along with the system under test and the measurement can be lengthy in order to remove the system errors (Hoefer et al 1975, Douville et al 1978, Easter 1975 and Slobodnik et al 1994). In this section time domain measurement will be used to validate the models. A TDR measurement is performed for each transmission line interconnection with a discontinuity at the center of the line. The model is extracted using either of the three methods described above and virtual TDR is performed on the model using SPICE circuit simulator. Results from simulation will be shown together with
actual TDR measurement for comparison. Time domain measurement has the advantages that it is easier to perform, and single TDR measurement will encompass a wide range of frequency within the spectrum of the step pulse.

3.6.2 Field Solution Method Modeling Example

Figure 3.12A shows the construction details of the via to be modeled. The electromagnetic field solver software is Maxwell 3D Parameter Extractor\(^1\). Figure 3.12B is the model of the via used for in the FEM. Observe that in the figure the top and bottom pads were removed to simplify the discretization process of the problem region. It is difficult to represent a three dimensional object of very small thickness effectively using tetrahedral elements, hence the ground planes are declared as Dirichlet boundary condition instead with potential set to zero. The FEM field solver software determines the equivalent lumped capacitance and inductance of the discontinuity from excess charge and flux. Symmetrical T and π models are created and circuit simulator is used to verify the models as compared to TDR measurement. Detail of the TDR verification is described in Section 3.5.3. The schematic of the circuit for circuit simulator is shown in Figure 3.12C. It will be used throughout the section for comparison between TDR measurement and simulated results.

![Cross sectional view of the via under measured.](image)

\(^1\) Maxwell is a trademark of Ansoft Inc.
Figure 3.12B - Model of via for FEM software.

Figure 3.12C - Schematic for circuit simulator.

Figure 3.12D shows the results of simulation. The model used is depicted as Model A. Observe that discrepancy in time is due to the assumption that stray capacitance and inductance are lumped. By distributing the lumped capacitance and inductance a better approximation can be attained.
3.5.3 TDR Modeling Examples

The incident step source for TDR measurement can be negative or positive going steps. However in all measurements only positive going step is used. The shape of the incident source is determined by sampling the reflected waveform taken from the end of a customized 40GHz probe, with the other end of the transmission line being left opened or shorted. Usually shorted termination is preferred, as it is very difficult to obtain a good opened termination. The software IPA310\(^2\) is used to derive the approximate lossless impedance profile for the interconnection based on the method in Section 3.3. The impedance profile is then partitioned and segments of equivalent transmission line are assigned to these partitioning. The characteristic impedance and propagation delay in each segment of a transmission line is the average value within its corresponding partitioning. A second TDR measurement is then performed to validate the reflected voltage waveform between circuit simulator and measurement. The circuit simulator software used is PSPICE\(^3\). First example is modeling of a 90 degrees stripline bend on a FR4 dielectric PCB. Figure 3.13A to Figure 3.13B shows the TDR response and the impedance profile extracted for a bend in stripline. In all validation measurement the other end of the transmission line is terminated with 50Ω termination. The schematic of the simulation circuit is similar to the schematic in Figure 3.12C.

\(^2\) IPA310 is a trademark of Tektronix Inc.
\(^3\) PSPICE is the trademark of Microsim Inc.
Reflection of incident step from shorted probe, $t_r = 45\text{ps}$

Reflection from bend

**Figure 3.13A** - Shape of incident waveform (probe shorted) and reflection of the bend.

* The effective impedance in each segment is the average within the segment
From Figure 3.13C, it is obvious that the segmented transmission line model provides adequate accuracy in modeling discontinuity. One reason the LC model provides erroneous result is the LC equivalent is only applicable for low frequencies, or for step transition rate of more than 500ps. The second example models a via discontinuity on FR4 PCB dielectric using the physical model in Figure 3.12A. Figure 3.14A is the impedance profile of the via and its equivalent model. Comparison of measured and simulated TDR waveform is shown in Figure 3.14B. An important observation from Figure 3.14A is that stray capacitance between the via and ground planes is the dominant effect, which causes the reduction in effective impedance.
**Figure 3.14A** - Impedance profile of DR31 via and partitioning for equivalent circuit computation.

**Figure 3.14B** - Comparison between measured and simulated TDR waveform of DR31 via using segmented transmission line model.
The third example is a ground plane gap in a stripline. The structure of the ground plane gap is shown in Figure 3.15A. Impedance profile and comparison between measurement and simulation results are in Figure 3.15B and 3.15C respectively. The important observation here is stray inductance becomes dominant in the presence of gap in planes, causing the increase of effective impedance seen.

Figure 3.15A - Construction of the ground plane gap.

* The effective impedance in each segment is the average within the segment
Figure 3.15B - Impedance profile of ground plane gap and partitioning for equivalent circuit computation.

Figure 3.15C - Comparison between measured and simulated result of ground plane gap using segmented transmission line model.

3.5.4 S-Parameters Measurement Modeling Example

The last example illustrates extraction of equivalent circuit for discontinuity in frequency domain using optimization method as outlined in the second approach of Section 3.4. The objective of this modeling example is to estimate the equivalent circuit model for the transition from coaxial cable to PCB trace. This transition is normally achieved using SMA to PCB adapter. Two SMA to PCB adapters are connected back to back into a FR4 PCB with built in ground plane as in Figure 3.16. S-parameters of this discontinuity is measured from 0.04GHz to 5.00GHz. Since this is a symmetrical discontinuity, \( L_1 = L_2 = L \) and \( S_{11} = S_{22} \).

The theoretical S-parameters are derived using equations (3.27a) and (3.27c):

\[
S_{11}(s) = \frac{\left(s^2 L^2 + \frac{2L}{c} - Z_o^2\right)s}{s^3 L^2 + 2Z_o L s^2 + \left(\frac{2L}{c} + Z_o^2\right)s + \frac{2Z_o}{c}} \tag{3.32a}
\]

\[
S_{12}(s) = \frac{2Z_o}{s^3 L^2 C + 2Z_o LC s^2 + \left(2L + Z_o^2 C\right)s + 2Z_o} \tag{3.32b}
\]

Using weighting constants of unity, the objective function now becomes:
\[
F( L, C ) = \sum_r \left[ \text{Re} S_{11}( j \omega_r ) - \text{Re} S_{111} \right]^2 + \sum_r \left[ \text{Re} S_{12}( j \omega_r ) - \text{Re} S_{121} \right]^2
+ \sum_r \left[ \text{Im} S_{11}( j \omega_r ) - \text{Im} S_{111} \right]^2 + \sum_r \left[ \text{Im} S_{12}( j \omega_r ) - \text{Im} S_{121} \right]^2
\]

(3.33)

The Method of Steepest Descent is used to minimize F(L,C). Initial guess values for L and C are assigned as L_0 and C_0. Experience has shown that it is difficult to apply equation (3.29a) to (3.29c) directly as L and C are small ( < 10^-9 ) and partial differentiation of F(L,C) with respect to L or C will become very large and highly unstable. For instance:

\[
\frac{\partial F( L = 2.1 \times 10^{-9}, C = 3.1 \times 10^{-12} )}{\partial L} = 5.1684 \times 10^9
\]

Such large partial differentiation value is highly undesirable and could cause the iteration to oscillate wildly. Thus instead of differentiating with respect to L and C, secondary variables x and y are introduced and L, C are related to x and y by:

L(x) = L_0 + x \Delta L \quad (3.34a)

C(y) = C_0 + y \Delta C \quad (3.34b)

where \( \Delta L \) and \( \Delta C \) are suitable chosen constants. In this example the following parameters are used:

L_0 = 1.0nH \quad \Delta L = 0.5nH

C_0 = 1.0pF \quad \Delta C = 0.5pF

The aim now is to find values of x and y which will minimize F(L,C). Thus the Method of Steepest Descent is applied, letting the initial guess values of x and y be \( x_0 = 0 \) and \( y_0 = 0 \). The next iterative solution for x and y will be:

\[
x_1 = x_0 - k_1 \frac{\partial F}{\partial L} \bigg|_{L(x_0), C(y_0)} \quad (3.35a)
\]

\[
x_1 = x_0 - k_1 \frac{\partial F}{\partial C} \bigg|_{L(x_0), C(y_0)} \quad (3.35b)
\]

where \( k_i, i = 0,1,2...n \) are positive constants called the local bounds. The local bound \( k_i \) is initially taken as \( k_0 = k_1 = 0.6 \) and is adjusted in every iteration based on successive difference between the objective function. Usually the local bound
will decrease as the iteration procedure proceeds to higher iterations. The rational is that the approximate solution will become closer to the actual solution after each iteration. Thus the local bound must be decreased to prevent oscillation around the actual solution. In this example the local bound is set as:

\[
k = P \text{ if } P > 0.25
\]

\[
0.25 \text{ otherwise.}
\]

with

\[
P = \frac{F(L(x_j), C(y_j)) - F(L(x_{j-1}), C(y_{j-1}))}{F(L(x_j), C(y_j))}
\]

where \( j = 0,1,2...n \). The iterative procedure will be stopped once the required accuracy has been achieved. To check for strong minimum, the Hessian matrix of the objective function can be examined as in Section 3.4. In this example all the differentiation operations are performed numerically e.g. using perturbation method. Iteration is stopped after seven loops, and the results are:

\[L = 2.352 \text{nH}\]

\[C = 2.714 \text{pF}\]

The simulated and measured \(S_{11}\) and \(S_{21}\) are compared in Figure 3.17A and Figure 3.17B for magnitude and phase, respectively. Good agreement is seen in this case up to 4.0GHz. Equivalent circuit of the coaxial to PCB trace discontinuity through SMA adapter is shown in Figure 3.18.

- Setup of measurement and the discontinuity.
In the event when more complicated equivalent circuit is used, it might be difficult to derived analytically the theoretical S-parameters. Under this condition the equivalent discontinuity model can be inserted into a circuit as in Figure 3.12C. Frequency domain simulation is then performed for a range of frequency from 400MHz to 25GHz using PSPICE. The simulated S-parameters are computed from the complex voltages and currents at both ports of the equivalent circuit using the original definition of S-parameters:

\[
S_{11} = (V_1 - Z_o I_1)(V_1 + Z_o I_1)^{-1} \quad (3.37a)
\]

\[
S_{21} = (V_2 - Z_o I_2)(V_1 + Z_o I_1)^{-1} \quad (3.37b)
\]
Figure 3.17A - Comparison of magnitude and phase for measured and theoretical $S_{11}$. 
Figure 3.17B - Comparison of magnitude and phase for measured and theoretical $S_{21}$.

Figure 3.18 - Equivalent circuit of coaxial to PCB trace discontinuity via SMA adapter.