Appendix 2 – Courant-Friedrichs-Lewy Stability Criterion

Much like the differential form of Maxwell's equation (3.2.5a)-(3.2.5d), we could also decouple the update equations for E and H field components as given by (3.3.4a)-(3.3.4d). This will be shown for E_z field component. Assuming the conductivity σ to be zero. Applying backward-difference operator at *y*-axis (δ_y^-) to (3.3.4a):

$$\delta_y^-\left(\frac{H_{x(i,j,k)}^{n+\frac{1}{2}}-H_{x(i,j,k)}^{n-\frac{1}{2}}}{\Delta t}\right)=-\frac{1}{\mu}\delta_y^-\left(\frac{E_{z(i,j+1,k)}^{n}-E_{z(i,j,k)}^{n}}{\Delta y}-\frac{E_{y(i,j,k+1)}^{n}-E_{y(i,j,k)}^{n}}{\Delta z}\right)$$

$$\Rightarrow \frac{H_{x(i,j,k)}^{n+\frac{1}{2}}-H_{x(i,j-1,k)}^{n+\frac{1}{2}}}{\Delta t\Delta y}-\frac{H_{x(i,j,k)}^{n-\frac{1}{2}}-H_{x(i,j-1,k)}^{n-\frac{1}{2}}}{\Delta t\Delta y}$$
$$=-\frac{1}{\mu}\left(\frac{E_{z(i,j+1,k)}^{n}-2E_{z(i,j,k)}^{n}+E_{z(i,j-1,k)}^{n}}{\Delta y^2}-\frac{\left(E_{y(i,j,k+1)}^{n}-E_{y(i,j-1,k+1)}^{n}\right)-\left(E_{y(i,j,k)}^{n}-E_{y(i,j-1,k)}^{n}\right)}{\Delta y\Delta z}\right) \tag{A2.1}$$

Applying backward-difference operator at *x*-axis (δ_x^-) to (3.3.4b):

$$\frac{H_{y(i,j,k)}^{n+\frac{1}{2}}-H_{y(i-1,j,k)}^{n+\frac{1}{2}}}{\Delta t\Delta x}-\frac{H_{y(i,j,k)}^{n-\frac{1}{2}}-H_{y(i-1,j,k)}^{n-\frac{1}{2}}}{\Delta t\Delta x}$$
$$=-\frac{1}{\mu}\left(\frac{\left(E_{x(i,j,k+1)}^{n}-E_{x(i-1,j,k+1)}^{n}\right)-\left(E_{x(i,j,k)}^{n}-E_{x(i-1,j,k)}^{n}\right)}{\Delta x\Delta z}-\frac{E_{z(i+1,j,k)}^{n}-2E_{z(i,j,k)}^{n}+E_{z(i-1,j,k)}^{n}}{\Delta x^2}\right) \tag{A2.2}$$

And finally applying backward-difference operator at time axis (δ_t^-) to (3.3.4f):

$$\delta_t^-\left(\frac{E_{z(i,j,k)}^{n+1}-E_{z(i,j,k)}^{n}}{\Delta t}\right)=\frac{1}{\varepsilon}\delta_t^-\left(\frac{H_{y(i,j,k)}^{n+\frac{1}{2}}-H_{y(i-1,j,k)}^{n+\frac{1}{2}}}{\Delta x}-\frac{H_{x(i,j,k)}^{n+\frac{1}{2}}-H_{x(i,j-1,k)}^{n+\frac{1}{2}}}{\Delta y}\right)$$

$$\Rightarrow \frac{E_{z(i,j,k)}^{n+1}-2E_{z(i,j,k)}^{n}+E_{z(i,j,k)}^{n-1}}{\Delta t^2}=\frac{1}{\varepsilon}\left(\frac{H_{y(i,j,k)}^{n+\frac{1}{2}}-H_{y(i-1,j,k)}^{n+\frac{1}{2}}}{\Delta x\Delta t}-\frac{H_{y(i,j,k)}^{n-\frac{1}{2}}-H_{y(i-1,j,k)}^{n-\frac{1}{2}}}{\Delta x\Delta t}\right)$$
$$-\frac{1}{\varepsilon}\left(\frac{H_{x(i,j,k)}^{n+\frac{1}{2}}-H_{x(i,j-1,k)}^{n+\frac{1}{2}}}{\Delta y\Delta t}-\frac{H_{x(i,j,k)}^{n-\frac{1}{2}}-H_{x(i,j-1,k)}^{n-\frac{1}{2}}}{\Delta y\Delta t}\right) \tag{A2.3}$$

Substituting (A2.1) and (A2.2) into (A2.3):

$$\frac{E^{n+1}_{z(i,j,k)}-2E^{n}_{z(i,j,k)}+E^{n-1}_{z(i,j,k)}}{\Delta t^2}=\frac{1}{\mu\varepsilon}\left(\begin{array}{l}\frac{E^{n}_{z(i+1,j,k)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i-1,j,k)}}{\Delta x^2}\\+\frac{E^{n}_{z(i,j+1,k)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i,j-1,k)}}{\Delta y^2}\end{array}\right)$$
$$-\frac{1}{\mu\varepsilon}\left(\begin{array}{l}\frac{\left(E^{n}_{y(i,j,k+1)}-E^{n}_{y(i,j-1,k+1)}\right)-\left(E^{n}_{y(i,j,k)}-E^{n}_{y(i,j-1,k)}\right)}{\Delta y\Delta z}\\+\frac{\left(E^{n}_{x(i,j,k+1)}-E^{n}_{x(i-1,j,k+1)}\right)-\left(E^{n}_{x(i,j,k)}-E^{n}_{x(i-1,j,k)}\right)}{\Delta x\Delta z}\end{array}\right) \tag{A2.4}$$

The second term on the right-hand side of (A2.4) can be written as:

$$\frac{\left(E^{n}_{y(i,j,k+1)}-E^{n}_{y(i,j-1,k+1)}\right)-\left(E^{n}_{y(i,j,k)}-E^{n}_{y(i,j-1,k)}\right)}{\Delta y\Delta z}+\frac{\left(E^{n}_{x(i,j,k+1)}-E^{n}_{x(i-1,j,k+1)}\right)-\left(E^{n}_{x(i,j,k)}-E^{n}_{x(i-1,j,k)}\right)}{\Delta x\Delta z}$$
$$=\delta_z^+\delta_y^- E^n_{y(i,j,k)}+\delta_z^+\delta_x^- E^n_{x(i,j,k)}=\delta_z^+\left(\delta_y^- E^n_{y(i,j,k)}+\delta_x^- E^n_{x(i,j,k)}\right)$$
$$=\delta_z^+\left(\frac{E^{n}_{y(i,j,k)}-E^{n}_{y(i,j-1,k)}}{\Delta y}+\frac{E^{n}_{x(i,j,k)}-E^{n}_{x(i-1,j,k)}}{\Delta x}\right)=\delta_z^+\left(-\frac{E^{n}_{z(i,j,k)}-E^{n}_{z(i,j,k-1)}}{\Delta z}\right)$$
$$=-\frac{E^{n}_{z(i,j,k+1)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i,j,k-1)}}{\Delta z^2}$$

Note that equation (3.3.7) is used to get the final expression above. Finally putting this result into (A2.4), the finite-difference wave equation for E_z field component is obtained.

$$\frac{E^{n+1}_{z(i,j,k)}-2E^{n}_{z(i,j,k)}+E^{n-1}_{z(i,j,k)}}{\Delta t^2}=\frac{1}{\mu\varepsilon}\left(\begin{array}{l}\frac{E^{n}_{z(i+1,j,k)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i-1,j,k)}}{\Delta x^2}\\+\frac{E^{n}_{z(i,j+1,k)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i,j-1,k)}}{\Delta y^2}\\+\frac{E^{n}_{z(i,j,k+1)}-2E^{n}_{z(i,j,k)}+E^{n}_{z(i,j,k-1)}}{\Delta z^2}\end{array}\right) \tag{A2.5}$$

A similar procedure can be used to derive the finite-difference wave equation for E_x and E_y field components. The wave equations can be solved independently, much like the continuous case. It is not necessary to obtain the finite-difference wave equation for H field components, as the H field components can be determined from (3.3.4a)-(3.3.4c) once the E field components are known. Since (A2.5) and similar equations for E_x and E_y field components are derived from the update equations, the two sets of equations are equivalent.

Discrete Fourier Transform (DFT) can be applied to the spatial variables of the E_z field component. For three-dimensional space, the following transform pair can be defined (Oppenheim 1989, Strikwerda 1989):

$$\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)=\sum_{r=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}E_{z(p,q,r)}^n e^{-j\left(\xi_x p\Delta x+\xi_y q\Delta y+\xi_z r\Delta z\right)}\Delta x\Delta y\Delta z \qquad \text{(A2.6a)}$$

$$E_{z(p,q,r)}^n=\frac{1}{(2\pi)^{3/2}}\int_{-\frac{\pi}{\Delta z}}^{\frac{\pi}{\Delta z}}\int_{-\frac{\pi}{\Delta y}}^{\frac{\pi}{\Delta y}}\int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}}\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)e^{j\left(\xi_x p\Delta x+\xi_y q\Delta y+\xi_z r\Delta z\right)}d\xi_x d\xi_y d\xi_z \qquad \text{(A2.6b)}$$

In (A2.6a) and (A2.6b), *p*, *q* and *r* are the integers for space grid instead of the usual *i*, *j* and *k*. This is to avoid confusion with the imaginary number $j=\sqrt{-1}$. The difference between (A2.6a) and (A2.6b) with the DFT encountered in digital signal processing are the multipliers Δx, Δy and Δz in the transform and inverse pairs. This is because in signal processing, the samples are thought to space one unit apart. In FDTD the sample as spaced Δx, Δy and Δz units apart in the space grid. We could look upon $\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)e^{j\left(\xi_x p\Delta x+\xi_y q\Delta y+\xi_z r\Delta z\right)}$ as one of the sinusoidal components for $E_{z(p,q,r)}^n$. Note that $\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)$ can be a complex value. Since superposition principle is valid for (A2.5), let us substitute $\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)e^{j\left(\xi_x x+\xi_y y+\xi_z z\right)}$ into (A2.5). Upon eliminating the common terms, this yields:

$$\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)\left[\frac{e^{j\xi_x}-2+e^{j\xi_x}}{\Delta x^2}+\frac{e^{j\xi_y}-2+e^{j\xi_y}}{\Delta y^2}+\frac{e^{j\xi_z}-2+e^{j\xi_z}}{\Delta z^2}\right]$$
$$=\mu\varepsilon\frac{\hat{E}_z^{n+1}\left(\xi_x,\xi_y,\xi_z\right)-2\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)+\hat{E}_z^{n-1}\left(\xi_x,\xi_y,\xi_z\right)}{\Delta t^2} \qquad \text{(A2.7)}$$

Let $c=\frac{1}{\sqrt{\mu\varepsilon}}$ and $D_n=\hat{E}_z^n\left(\xi_x,\xi_y,\xi_z\right)$ (A2.8)

Note that in (A2.8), the superscript *n* represents time-step and not power. D_n is a sequence. Rearranging and using Euler's identity in (A2.7) (Hockanson 1994):

$$D_{n+1}-2D_n\left[1-2\left(c\frac{\Delta t}{\Delta x}\right)^2\sin^2\frac{\xi_x}{2}-2\left(c\frac{\Delta t}{\Delta y}\right)^2\sin^2\frac{\xi_y}{2}-2\left(c\frac{\Delta t}{\Delta z}\right)^2\sin^2\frac{\xi_z}{2}\right]+D_{n-1}=0 \qquad \text{(A2.9)}$$

This can be written in a more compact form as:

$$D_{n+1} - 2D_n a + D_{n-1} = 0$$
$$a = 1 - 2\left(c\tfrac{\Delta t}{\Delta x}\right)^2 \sin^2 \tfrac{\xi_x}{2} - 2\left(c\tfrac{\Delta t}{\Delta y}\right)^2 \sin^2 \tfrac{\xi_y}{2} - 2\left(c\tfrac{\Delta t}{\Delta z}\right)^2 \sin^2 \tfrac{\xi_z}{2} \qquad \text{(A2.10)}$$

Notice that (A2.10) is just a linear second order difference equation (James 1993, Allen and Isaacson, 1998). A solution for (A2.10) is:

$$D_n = q^n \qquad \text{(A2.11)}$$

In (A2.11) *q* can be a real or complex value and *n* represents power. Therefore,

$$\hat{E}_z^n\left(\xi_x, \xi_y, \xi_z\right) e^{j\left(\xi_x x + \xi_y y + \xi_z z\right)} = q^n e^{j\left(\xi_x x + \xi_y y + \xi_z z\right)} \qquad \text{(A2.12a)}$$

$$q^2 - 2qa + 1 = 0 \qquad \text{(A2.12b)}$$

For the E_z to be bounded, $|q| < 1$. Otherwise as *n* increases $\hat{E}_z^n\left(\xi_x, \xi_y, \xi_z\right)$ will diverge and from (A2.6b) E_z field components will also diverge. The roots of *q* in (A2.12b) must be such that:

$$q = \tfrac{1}{2}\left(a \pm \sqrt{a^2 - 1}\right) \qquad \text{(A2.13)}$$

We observe that $|q| < 1$ if we require that $a^2 - 1 < 0$. This can be expanded as:

$$a^2 < 1 \Rightarrow -1 < a < 1$$
$$\Rightarrow \ -1 < 1 - 2\left(c\tfrac{\Delta t}{\Delta x}\right)^2 \sin^2 \tfrac{\xi_x}{2} - 2\left(c\tfrac{\Delta t}{\Delta y}\right)^2 \sin^2 \tfrac{\xi_y}{2} - 2\left(c\tfrac{\Delta t}{\Delta z}\right)^2 \sin^2 \tfrac{\xi_z}{2} < 1 \qquad \text{(A2.14)}$$

Equation (A2.14) can be rearranged as:

$$\left(c\Delta t\right)^2 \left[\frac{\sin^2 \frac{\xi_x}{2}}{\Delta x^2} + \frac{\sin^2 \frac{\xi_y}{2}}{\Delta y^2} + \frac{\sin^2 \frac{\xi_z}{2}}{\Delta z^2}\right] < 1 \qquad \text{(A2.15)}$$

The limiting value allowed for the time-step is obtained by letting the sine functions take their maximum value of unity. The FDTD scheme will then be stable given:

$$\Delta t < \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}} \qquad \text{(A2.16)}$$

This proves the CFL Stability Criterion. Observe that from (A2.6a), because of the infinite samples required in the summation, it is implicitly assumed that the computational region to be infinite or the absorbing boundary condition is ideal. The usage of Discrete Fourier Transform (DFT) in deriving the stability condition for finite-difference is the essence of the von Neumann approach. It could be shown that when the conductivity σ of the dielectric is not zero, similar criterion as in (A2.16) is also obtained (Pereda 1998).