

## **Appendix A Concept of Partial Inductance**

### **A.1 Introduction**

In using lumped circuit approximation to determine the inductance of a conductor loop, care must be taken to ensure that the size of the conductor is small ( $\leq 0.2$ ) as compared to the minimum wavelength likely to encounter in the system. If the size of the conductor loop is considerable with respect to shortest wavelength, we could take two steps :

- Divide the conductor into a number of segments and
- Treat the conductor with ground return as transmission line with varying characteristic impedance as a function of distance

Analysis of step one above will yield a lumped approximation for a conductor loop. It involves the use of partial inductance concept. General inductance definition involves a closed loop current. Even when the conductor is not closed, an imaginative closed path is assumed as in the case of the inductance definition for a two conductors transmission line system. This inductance definition is generally called loop inductance. Partial inductance is defined as the inductance of a single conductor with respect to infinity as the reference for return current. This concept can be applied to conductor with complex shape. It is beneficial to consider the analysis of partial inductance as this idea can be generalized to collection of conductors. A good example would be a socket or integrated circuit package with a collection of pins and bond wires. Results from partial inductance analysis will aid in systematic generation of equivalent circuit for component with a lots of pins/connection such as integrated circuit socket, package and edge connectors.

### **A.2 Analysis of Inductance in Multiconductor Environment**

Generally there are two approaches at our disposal for defining lumped parameter such as inductance. We could follow the field theory definition using Faraday's second law or the alternative through energy consideration. Such considerations

are also true for capacitance definition. Field theory approach stems largely from the work of Ruehli 1972 and it will be used to define partial inductance in Section A.3. Definition of partial inductance in terms of energy consideration will be given in Section A.4.

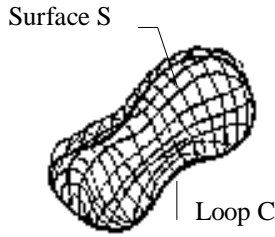
Loop inductance is defined as the ratio of flux linkage to the total current flowing in the closed loop S (Figure A.1).

$$L_{Loop} = \frac{\Psi}{I} \quad (A.1a)$$

$$\Psi = \iint_S \mathbf{B} \cdot \mathbf{ds} \quad (A.1b)$$

From Faraday's second law and Lenz's Principle :

$$V_{induced} = -L \frac{dI}{dt} \quad (A.2)$$



**Figure A.1** - Closed loop and surface.

Upon using Stoke's theorem and equation (A.1b) :

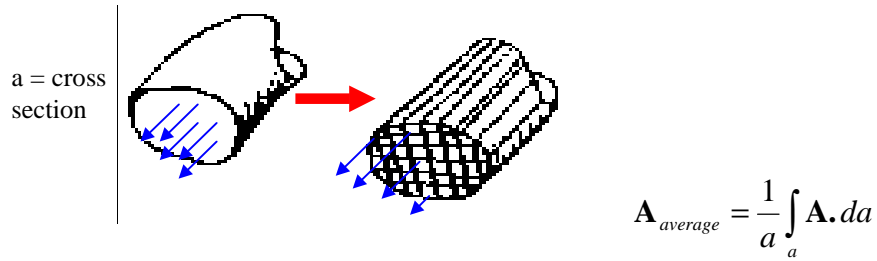
$$\Psi = \iint_S \mathbf{B} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{ds} = \oint_C \mathbf{A} \cdot \mathbf{dl} \quad (A.3a)$$

where  $\mathbf{A}$  the static magnetic vector potential is given by :

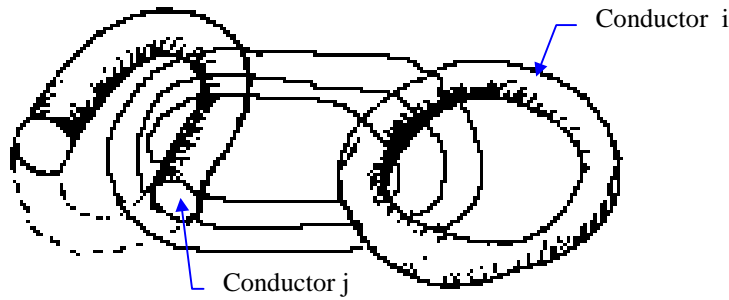
$$\mathbf{A} = \frac{\mu_r \mu_o}{4\pi} \iiint_V \frac{\mathbf{J} dv}{r} \quad (A.3b)$$

Introducing the subscript notation, let  $\mathbf{J}_i$  be the current density on conductor loop  $i$  and  $\Psi_{ij}$  be the flux link to conductor loop  $i$  due to current flowing in conductor loop  $j$ . The above derivation is only valid for conductor of infinitesimally small cross section. For a conductor with finite cross section, the average vector

potential across the cross section of the conductor is considered. The voltage at the end of a conductor is expected to be the average between the maximum induced potential through the maximum loop size and the minimum induced potential through the minimum loop size. This concept is illustrated in Figure A.2 and Figure A.3.



**Figure A.2** - Dividing a conductor with finite cross section into strip elements.



**Figure A.3** - A two loop system.

Hence the average flux linkage in loop i due to current in loop j ( $I_j$ ) is :

$$\Psi_{ij(average)} = \langle \Psi_{ij} \rangle = \frac{1}{a_i} \oint_{C_i} \int_{a_i} \mathbf{A}_{ij} \cdot d\mathbf{l}_i da_i \quad (\text{A.4})$$

Combining equation (A.3b) and equation (A.4) :

$$\langle \Psi_{ij} \rangle = \frac{\mu_r \mu_o}{4\pi a_i} \oint_{C_i} \left( \int_V \mathbf{J}_j dv \right) \cdot d\mathbf{l}_i da_i \quad (\text{A.5})$$

Assuming both loop i and loop j to have uniform cross section  $a_i$  and  $a_j$ , and uniform uniform current density (consider static condition,  $\mathbf{J} = (I_i/a_i) \mathbf{j}$ ) :

$$\langle \psi_{ij} \rangle = \frac{\mu_r \mu_o}{4\pi a_i} \oint_{C_i} \int_{C_j} \left[ \frac{I_j da_j \mathbf{dl}_j}{a_j} \right] \bullet \mathbf{dl}_i da_i \quad (\text{A.6a})$$

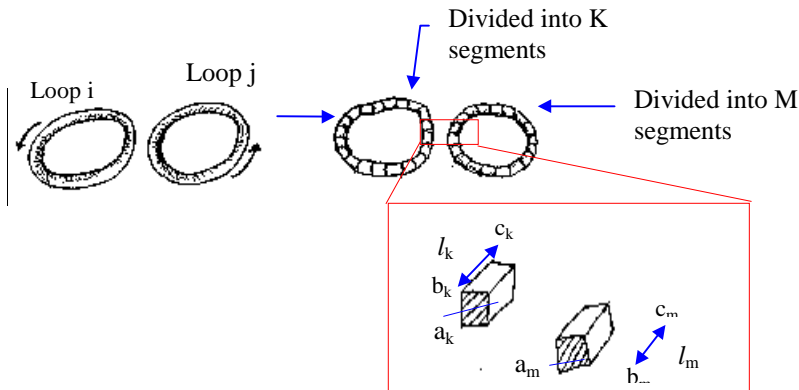
$$\langle \psi_{ij} \rangle = \frac{\mu_r \mu_o I_j}{4\pi a_i a_j} \int_{a_i} \int_{a_j} \left[ \oint_{C_i} \oint_{C_j} \frac{\mathbf{dl}_i \bullet \mathbf{dl}_j}{r_{ij}} \right] da_i da_j \quad (\text{A.6b})$$

For infinitely thin wires, equation (A.6b) reduces to the well - known Neumann formula :

$$\langle \psi_{ij} \rangle = \frac{\mu_r \mu_o I_j}{4\pi a_i a_j} \left[ \oint_{C_i} \oint_{C_j} \frac{\mathbf{dl}_i \bullet \mathbf{dl}_j}{r_{ij}} \right] \quad (\text{A.7})$$

### A.3 Partial Inductance Definition Through Field Theory

In dividing large conductor loops are subdivided into cascaded discrete segments, all segments that belongs to the same conductor are assumed to be of uniform cross section (Figure A.4).



Segmentation of conductor loops.

As shown in Figure A.4, a continuous loop is broken into finite number of segments, each segment is represented by simple geometrical structure with rectangular or circular cross sections. The mutual inductance between conductor loop i and conductor loop j can be expressed as :

$$L_{ij} = \frac{\langle \psi_{ij} \rangle}{I_j} \cong \sum_{k=1}^K \sum_{m=1}^M \frac{\mu_r \mu_o}{4\pi} \cdot \frac{1}{a_k a_m} \int_{a_k}^{c_k} \int_{a_m}^{c_m} \int_{b_k}^{c_k} \int_{b_m}^{c_m} \frac{d\mathbf{l}_k \bullet d\mathbf{l}_m}{r_{km}} da_k da_m \quad (\text{A.8})$$

From equation (A.8) the expression within the double summation operator is called the partial mutual inductance between segment k and segment m of loop i and loop j. For instance partial inductance between segment k and segment m is :

$$L_{pkm} = \frac{\mu_r \mu_o}{4\pi} \cdot \frac{1}{a_k a_m} \int_{a_k}^{c_k} \int_{a_m}^{c_m} \int_{b_k}^{c_k} \int_{b_m}^{c_m} \frac{d\mathbf{l}_k \bullet d\mathbf{l}_m}{r_{km}} da_k da_m \quad (\text{A.9})$$

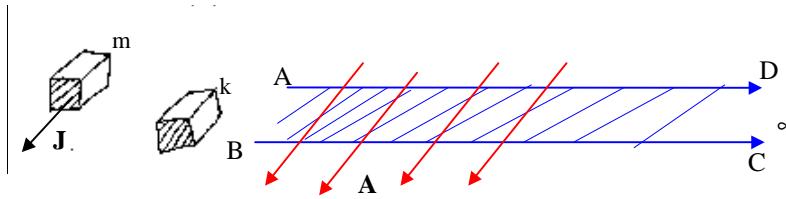
If  $\theta_{km}$  is the angle between vector  $d\mathbf{l}_k$  and  $d\mathbf{l}_m$  in Figure A.4, then (A.8) can be summarized as :

$$L_{ij} = \sum_{k=1}^K \sum_{m=1}^M S_{km} L_{pkm} \quad (\text{A.10a})$$

Where :

$$S_{km} = \begin{cases} 1, & \text{if } \theta_{km} < \frac{\pi}{2} \\ -1, & \text{if } \theta_{km} > \frac{\pi}{2} \\ 0, & \text{if } \theta_{km} = 0 \end{cases} \quad (\text{A.10b})$$

When  $k = m$ ,  $L_{pkk} = L_{pmm}$  is called partial self inductance while  $L_{pkm}$  is called the partial inductance between element k and element m. Note some interesting implication out of the previous analysis, equations (A.3a ) and (A.3b) indicates that magnetic vector potential  $\mathbf{A}$  is always parallel to the direction of current flow as shown in Figure A.5.



**Figure A.5** - Physical meaning of partial mutual inductance.

$$\Psi_{ABCD} = \oint_C \mathbf{A} \cdot d\mathbf{l} = \int_{AB} \mathbf{A} \cdot d\mathbf{l} + \int_{BC} \mathbf{A} \cdot d\mathbf{l} + \int_{CD} \mathbf{A} \cdot d\mathbf{l} + \int_{DA} \mathbf{A} \cdot d\mathbf{l} \quad (\text{A.11})$$

$$\Psi_{ABCD} = \iint_{ABCD} \mathbf{B} \cdot d\mathbf{S} = \int_{AB} \mathbf{A} \cdot d\mathbf{l}$$

Since there is no contribution from magnetic vector potential which is perpendicular to side BC and CD. From equation (A.9):

$$L_{pl(k,m)} = \frac{\mu_r \mu_o}{4\pi} \cdot \frac{1}{a_k a_m} \int_{a_k}^{c_k} \int_{a_m}^{c_m} \int_{b_k}^{b_m} \frac{d\mathbf{l}_k \cdot d\mathbf{l}_m}{r_{km}} da_k da_m = \frac{\Psi_{ABCD}}{I} \quad (\text{A.12})$$

From equation (A.12), it is obvious that partial mutual inductance between two conductors is the ratio of flux linkage from one conductor to infinity to the current in the other conductor.

#### A.4 Partial Inductance Definition Through Energy Consideration

It is known that for loop inductance, the energy stored in the magnetic field is given by :

$$W_{magnetic} = \frac{1}{2} LI^2 = \frac{1}{2\mu} \iiint_V \mathbf{B} \cdot \mathbf{B}^* dv \quad (\text{A.13})$$

Consider equation (A.6b) again, the product of LI gives the total flux linkage of a current loop. Hence from equation (A.3a):

$$W_{magnetic} = \frac{1}{2} I \oint_C \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint_C \mathbf{A} \cdot \mathbf{I} dl$$

$$\Rightarrow W_{magnetic} = \frac{1}{2} \oint_C \mathbf{A} \cdot \iint_S \mathbf{J} ds dl = \frac{1}{2} \iiint_V \mathbf{A} \cdot \mathbf{J} dv \quad (\text{A.14})$$

In equation (A.14) the volume  $V$  refers to the volume of the three dimensional conductor loop. When more than one conductor loops are present, stored energy due to magnetic flux from loop  $j$  to loop  $i$  is :

$$\frac{dW}{dt} = L_{ij} I_i \frac{dI_j}{dt} \quad (\text{A.15a})$$

$$\text{where } W = L_{ij} I_i I_j \quad (\text{A.15b})$$

Following the same procedure as of above :

$$W = I_i \oint_{C_i} \mathbf{A}_j \cdot d\mathbf{l} = \iint_{S_i} J_i ds \cdot \oint_{C_i} \mathbf{A}_j \cdot d\mathbf{l}_i = \iiint_{V_i} \mathbf{A}_j \cdot \mathbf{J}_i dv \quad (\text{A.15c})$$

$$\text{Thus } L_{ij} = \frac{\iiint_{V_i} \mathbf{A}_j \cdot \mathbf{J}_i dv}{I_i I_j} \quad (\text{A.16a})$$

for mutual loop inductance between loops and :

$$L_{ii} = \frac{\iiint_{V_i} A_i \cdot J_i dv}{I_i^2} \quad (\text{A.16b})$$

for self loop inductance from energy consideration. By dividing the loop of interest into K segments, the general loop inductance formula becomes :

$$L_{ij} = \frac{\sum_{i=1}^K \iiint_{V_i} \mathbf{A}_j \cdot \mathbf{J}_i dv}{I_i I_j} \quad (\text{A.17})$$

If the source for the vector magnetic potential  $\mathbf{j}$  is divided in M segments, then  $\mathbf{A}_j$  can be written as :

$$\mathbf{A} = \sum_{j=1}^M \mathbf{A}_j \quad (\text{A.18})$$

Combining equations (A.17) and (A.18) into equation (A.16b) :

$$L_{ij} = \sum_{k=1}^K \sum_{m=1}^M \frac{\iiint_{k_i} \mathbf{A}_m \cdot \mathbf{J}_{k_i} dv}{I_i I_j} = \sum_{k=1}^K \sum_{m=1}^M L_{pkm} \quad (\text{A.19})$$

Where the terms  $L_{pkm}$  is the partial inductance between element k in loop i and element m in loop j.

## A.5 Summary

Both equations (A.9) and (A.19) can be used to derived the partial inductance expression for an array of complicated conductors. Partial inductances are extremely useful when we need to describe the inductance of a portion of a complete current loop. Its advantage is seen when we have a piece of conductor and we do not know the actual flow of the return current to complete the current loop. To ascertain the effect of the conductor, we would need to find the partial self inductance of the conductor and the partial mutual inductances between the

conductor and other conductors in the vicinity. The representation of conductors using partial inductances is only valid when length of the conductor is much shorter than the shortest wavelength encountered in the system, in other words retardation effect can be neglected and the conductor can be thought of as a lumped element.